ON THE THEORY OF CONVECTIVE PHENOMENA

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Abstract

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ON THE THEORY OF CONVECTIVE PHENOMENA

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Many questions in the theory of stability of mechanical equilibrium can be reduced to the following general problem: for small deviations of some system parameter x from the equilibrium value $x = 0$, two forces of different nature arise: one, stabilizing, tends to decrease the deviation; the other, destabilizing, tends to increase it. Relaxation of both forces can be described according to Maxwell' s scheme, expressed by the equations

$$\ddot{x} = f_1 + f_2; \qquad df_1/dt + \delta_1 f_1 = -\Omega^2 dx/dt; \\ df_2/dt + \delta_2 f_2 = \chi^2 dx/dt. \tag{1}$$

(in rapid processes $f_{1,2} \sim x$, and in slow ones $\sim e^{-\delta t}$). The decrements δ_1 and δ_2 are determined by the law of relaxation of the forces, while the frequency Ω and the increment χ are determined by solving the problem of small perturbations with f_1 and f_2 taken separately without relaxation. As a rule, the parameters $\delta_1, \delta_2, \Omega$, and χ in each particular case admit a simple estimate of order of magnitude. In the case of a continuous medium they are, generally speaking, functionals of the spatial structure of the perturbations.

Putting $f_1, f_2 \sim e^{pt}$, we obtain the characteristic equation

$$p^3 + a_1 p^2 + a_2 p + a_3 = 0 \tag{2}$$

with real coefficients $a_1 = \delta_1 + \delta_2; \quad a_2 = \delta_1\delta_2 + \Omega^2 - \chi^2, \quad a_3 = \Omega^2\delta_1 - \chi^2\delta_1$. According to the Hurwitz criterion, a necessary and sufficient condition for negativity of the real parts of the roots of the polynomial (2) with real coefficients is the simultaneous fulfillment of the inequalities $a_1 a_2 - a_3 > 0;\ a_3 > 0$.

Thus, the system under consideration proves to be stable if and only if

$$\chi^2 < \min\left\{\frac{\delta_1}{\delta_2}\left[(\delta_1+\delta_2)\delta_2+\Omega^2\right];\ \frac{\delta_2}{\delta_1}\Omega^2\right\}. \tag{3}$$

For a number of applications, the case is especially interesting when $\Omega^2 \gg \chi^2$, i.e., under rapid displacements the stabilizing force greatly exceeds the destabilizing one.

For a certain ratio of the decrements δ_1 and δ_2, the stability condition (3) may be violated. The increment of the resulting instability can be determined from the solution of the cubic equation (2). For $\delta_1, \delta_2 \ll \Omega$, the roots of this equation have the form

$$p_1 = -[\delta_2 - (\delta_1 - \delta_2)\chi^2/\Omega^2], \quad p_2, p_3 = -[\delta_1 - (\delta_2 - \delta_1)\chi^2/\Omega^2]/2 \pm i\Omega.$$

The system may prove unstable when the decrements δ_1 and δ_2 differ sharply. To accuracy up to small quantities of higher orders, we have: for $\delta_2 \ll \delta_1$

$$p_1 = -[\delta_2 - \delta_1\chi^2/\Omega^2], \qquad p_2, p_3 = -\delta_1/2 \pm i\Omega; \tag{4}$$

for $\delta_2 \gg \delta_1$

$$p_1 = -\delta_2, \qquad p_2, p_3 = -[\delta_1 - \delta_2\chi^2/\Omega^2]/2 \pm i\Omega.$$

Thus, when the stabilizing force relaxes much faster than the destabilizing one, the stability of the slow solution may be lost. In the opposite case of a rapidly relaxing destabilizing force, loss of stability under rapid oscillations is possible.

Let us consider two examples. As is known, the Earth' s mantle can be regarded as a liquid only very conditionally. However, the criterion for the onset of thermal convection in the mantle, as will be shown with the aid of (4), has the same form as in a liquid (see, for example, ([1]), § 56). The elastic forces arising in the mantle under shear deformations are, in some approximation, described by Maxwell' s law. If l is the characteristic size of the region under consideration, then $\Omega^2 \approx (2\pi)^2 u_\perp^2/l^2$, where $u_\perp$ is the transverse sound velocity. The decrement $\delta_1 = 1/\tau$, where τ is the relaxation time of the shear stresses.

The destabilizing force is the Archimedean force. For small displacements of a volume element of size $\sim l$, this force is $F \sim g\beta\rho l^3|\nabla T|x$, where g is the acceleration of gravity, β is the volume coefficient of thermal expansion, ρ is the density, and ∇T is the temperature gradient. Hence $\chi^2 = F/Mx = g\beta|\nabla T|$ (here $M \sim \rho l^3$ is the mass of the volume element). The relaxation time of the Archimedean force is determined by the time of thermal conductivity $\tau_T \sim l^2/\chi$ (χ is the thermal diffusivity coefficient).

In this case one may assume that $\Omega \gg \chi$ and, for sufficiently large l, $\delta_2 \ll \delta_1$. From (4) we obtain the following condition for the onset of convection:

$$g\beta l^3 \delta T/\chi u_\perp^2 \tau \approx g\beta l^3 \delta T/\chi\nu > C,$$

where $\delta T \sim l\nabla T$, and C is a certain dimensionless constant determined by the geometry of the problem. We have taken into account that, according to the theory of liquids ([2]), the kinematic viscosity is $\nu = u_\perp^2 \tau$. Thus, consideration of the problem of convective instability on the basis of the arguments developed above leads to the same results as the purely hydrodynamic solution: the onset of convection is determined by the value of the dimensionless Rayleigh number $g\beta l^3 \delta T/\chi\nu$. To determine the minimum value of C at which convection arises, a solution of the boundary-value problem is, of course, required.

Another example is the magnetic convective instability of the Earth' s magnetosphere. As is known, near the surface bounded by field lines, at an equatorial distance from the center of the Earth $\approx 4a$ (a is the radius of the Earth), there is a sharp drop in the pressure of the cold plasma ([3]). If there were no ionosphere, convective instability would then have to arise ([4]). The increment may be estimated from the following considerations. Let l be the width of the pressure-drop region p in the equatorial plane. Under adiabatic displacement of a flux tube to the point x, the external pressure $p(x)$ will differ from the internal pressure p_T by the amount $(\nabla p + \gamma p \nabla U/U)x$, where the function $U = -\int dl/H$ is the volume of a flux tube of unit magnetic flux, taken with the opposite sign. The difference between the external and internal magnetic fields is then $(\nabla p + \gamma p \nabla U/U)4\pi x/H$. Per unit volume of the displaced tube there will act a buoyancy force $F = -I\nabla H$, where I is the diamagnetic moment of a unit volume of matter,

$$I = (\nabla p + \gamma p \nabla U/U)x/H.$$

Hence

$$\chi^2 = F/\rho x = (\nabla p + \gamma p \nabla U/U)|\nabla H|x/H\rho.$$

In the region of a significant pressure drop, $\nabla p \approx p/l \gg \gamma p \nabla U/U \approx \gamma p/L$, which permits us to write

$$\chi^2 \approx \frac{p}{H\rho l}|\nabla H|. \tag{5}$$

The elastic force is due to the fact that the ends of the lines of force are frozen into the dense conducting ionosphere. If it is regarded as ideally conducting and rigid, the pressure drop proves stable ([4]). In reality, however, the ionosphere is not rigid ([5]). More significantly, apparently, it is not ideally conducting, and the decay time of currents in a layer of thickness $h \sim 100$ km, for a real ionospheric conductivity $\sigma \sim 10^8 \div 10^9$ sec^{-1},

$$\tau = 4\pi\sigma h^2/c^2 \approx 10^2 \div 10^3 \text{ sec.}$$

is relatively small.

The frequency of elastic oscillations of the field tube of force in the present case is determined by the Alfvén velocity $H/\sqrt{4\pi\rho}$ and by the length of the line of force L,

$$\Omega^2 = \pi H^2/\rho L^2, \tag{6}$$

where H and ρ, as in (5), should be taken near the summit of the line of force.

If diffusion of particles across the lines of force is neglected and $\delta_2 = 0$ is assumed, then from (4), (5), and (6) we obtain the following expression for the development time of the instability of large-scale troughs*

$$T = C' \frac{4\pi^2\sigma h^2 H^3 l}{c^2 L^2 |\nabla H| \, p} \approx C' \frac{4\pi^2\sigma h^2}{c^2} \frac{H^2 l}{pL}, \tag{7}$$

where we have put $\nabla H \approx H/L$. The dimensionless constant C' is determined by a more detailed account of the geometry. Apparently, C' differs from 1 by no more than an order of magnitude.

Thus, the development time of the convective instability with scale $\sim l$ should be $\sim 100 \div 1000$ days (if one sets $p \sim 10^{-10}$ dyn/cm^2, $H \sim 5 \cdot 10^{-3}$ gauss, $\sigma = 10^8 \div 10^9$ sec^{-1}). We note that this time increases very rapidly (as L^{-7}, where L is the distance from the Earth' s center in the equatorial plane) as L decreases. Therefore a sharp pressure drop appears quite natural.

At the same time it should be pointed out that convective mixing in no way affects the distribution of fast particles in the radiation belts, since the drift velocity of these particles exceeds the convection velocity by many orders of magnitude.

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CITED LITERATURE

1. L. D. Landau, E. M. Lifshitz, *Mechanics of Continuous Media*, Moscow, 1953.

2. Ya. I. Frenkel, *Kinetic Theory of Liquids*, Collected Selected Works, 3, Moscow-Leningrad, 1959.

3. K. I. Gringauz, V. G. Kurt, et al., DAN, 132, 1062 (1960).

4. B. B. Kadomtsev, *Problems of Plasma Theory*, 2, 1963.

5. I. P. Ivaneenko, V. P. Shabanskii, *Geomagnetism and Aeronomy*, 1, No. 6, 881 (1961).

* That is, troughs whose linear dimensions above the ionosphere are $\geq h$.

Note: Figure translations are in progress. See original paper for figures.