ABSOLUTES OF HOMOGENEOUS SPACES

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Abstract

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ABSOLUTES OF HOMOGENEOUS SPACES

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All spaces in this note are completely regular. The absolute* pX of a topological space X is the irreducible perfect extremally disconnected preimage of X. In this note it is proved that the absolute of a bicompact topological group G of weight τ is homeomorphic to pD^τ, where D^τ is the generalized Cantor discontinuum of weight τ, which gives an answer to a question posed by A. V. Arhangel'skii. From this it follows, by virtue of a theorem of E. G. Sklyarenko (4), that the absolute of every locally bicompact group is homeomorphic to the product $T \times pD^\tau$, where T is a discrete space. An analogous result is also true for factor spaces of locally bicompact groups. Further, topologically homogeneous absolutes of any local power are constructed, which gives an answer to another question of A. V. Arhangel'skii. It is proved that the absolute of any topological group decomposes into everywhere dense disjoint topologically homogeneous absolutes.

§ 1. **Absolutes of locally bicompact groups and their factor spaces.**

Lemma 1. Let $X = \lim_{\leftarrow}\{X_\alpha, \varphi^\alpha_\beta\}$ and $Y = \lim_{\leftarrow}\{Y_\alpha, \psi^\alpha_\beta\}$ be inverse limits of bicompacta with "onto"mappings. Further, let $\Phi = \{f_\alpha\} : \{X_\alpha, \varphi^\alpha_\beta\} \to \{Y_\alpha, \psi^\alpha_\beta\}$ be a mapping of the spectrum into the spectrum such that, for every α, the mapping $f_\alpha : X_\alpha \to Y_\alpha$ is an irreducible mapping of X_α onto Y_α. Then the limiting mapping $f = \lim_\alpha f_\alpha$ is an irreducible mapping of X onto Y.

Proof. Recall that $f = \lim_\alpha f_\alpha$ is defined as follows: if $x = \{x_\alpha\} \in X$, then $f(x) = \{f_\alpha(x_\alpha)\} = y$. We prove the irreducibility of f. For this it is sufficient to prove that, for any basic open set $\varphi_\alpha^{-1}(U_\alpha) \subset X$, where $U_\alpha \subset X_\alpha$ and $\varphi_\alpha : X \to X_\alpha$, there exists $y \in Y$ such that $f^{-1}y \subset \varphi_\alpha^{-1}(U_\alpha)$. Since the mapping $f_\alpha : X_\alpha \to Y_\alpha$ is irreducible, there exists $y_\alpha \in Y_\alpha$ such that $U_\alpha \supset f_\alpha^{-1}(y_\alpha) = F_\alpha$. Since for any point $x \in X$ we have $\psi_\alpha f(x) = f_\alpha \varphi_\alpha(x)$, it follows that

$$\varphi_\alpha^{-1} f_\alpha^{-1}(y_\alpha) = f^{-1}\psi_\alpha^{-1}(y_\alpha).$$

On the other hand, $\varphi_\alpha^{-1}(F_\alpha) \subset \varphi_\alpha^{-1}(U_\alpha)$. Therefore, taking an arbitrary point $y \in \psi_\alpha^{-1}(y_\alpha)$, we obtain that $f^{-1}(y) \subset \varphi_\alpha^{-1}(U_\alpha)$, as was required to prove.

Lemma 2. Let $X = \lim_{\leftarrow}\{X_\alpha, \varphi^\alpha_\beta\}$ be an inverse limit of bicompacta indexed by all transfinite numbers $< \omega(\tau)$, and suppose the following conditions are satisfied: 1) X_1 is a metrizable compactum without isolated points; 2) for any neighboring ordinal numbers α and $\alpha+1$, the mapping $\varphi_\alpha^{\alpha+1}$ is a locally trivial fibration with fiber K_α, which is a metrizable compactum, and $|K_\alpha| \geq 2$**; 3) if γ is a limiting po- cardinal number, then $X_\gamma = \lim\limits_{\leftarrow}\{X_\alpha, \varphi_\alpha^\beta\}$. In this case X is an irreducible image of D^τ.

Proof. Consider X_1. Since X_1 is a metrizable compactum without isolated points, by Theorem 7 of (5) there exists an irreducible mapping f_1 of the Cantor perfect set $C = D^{\aleph_0} = D^{\tau(1)}$ onto X_1. Suppose that for all ordinal numbers $\alpha < \gamma$ irreducible mappings "onto" $f_\alpha : D^{\tau(\alpha)} \to X_\alpha$ have been constructed and that for any $\beta < \alpha < \gamma$ we have $f_\beta \pi_\beta^\alpha = \varphi_\beta^\alpha f_\alpha$, if π_β^α is the projection of

$$D^{\tau(\alpha)} = \prod_{\zeta<\alpha} Z_\zeta$$

onto the face

$$D^{\tau(\beta)} = \prod_{\zeta<\beta} Z_\zeta,$$

where the Z_ζ are zero-dimensional metrizable compacta. If γ is a limit number, then, taking the inverse limit, by property 3) we obtain that $X_\gamma = \lim\limits_{\leftarrow}\{X_\alpha, \varphi_\beta^\alpha\}$. Further, applying Lemma 1, we obtain that $f_\gamma = \lim f_\alpha$ is an irreducible mapping

$$D^{\tau(\gamma)} = \prod_{\zeta<\gamma} Z_\zeta = \lim_{\leftarrow}\{D^{\tau(\alpha)}, \pi_\beta^\alpha\}$$

onto X_γ. If $\gamma = \alpha+1$, then consider X_α. By the inductive hypothesis there exists an irreducible mapping $f_\alpha : D^{\tau(\alpha)} \to X_\alpha$ such that for any $\beta < \alpha$ we have $f_\beta \pi_\beta^\alpha = \varphi_\beta^\alpha f_\alpha$. The mapping $\varphi_\alpha^{\alpha+1} : X_{\alpha+1} \to X_\alpha$ is a locally trivial fibration. This means that for each point $x \in X_\alpha$ there exists a neighborhood $V'(x)$ such that $X_{\alpha+1}$ over the neighborhood $V'(x)$ is the direct product $V'(x) \times K_\alpha$, and the mapping $\varphi_\alpha^{\alpha+1}$ is the projection of $V'(x) \times K_\alpha$ onto $V'(x)$. For each point $x \in X_\alpha$ consider a neighborhood $x \in V(x) \subseteq \overline{V(x)} \subseteq V'(x)$. By the bicompactness of

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* The definition and properties of the absolute, as well as the related notions of extremal disconnectedness, irreducibility, and perfection of mappings, are given in ($^{1-3}$). In particular, a mapping $f : X \to Y$ is irreducible if there is no proper closed subset $F \subset X$ such that $f(F) = Y$.

** $|X|$ is the cardinality of the set X.

X_α there exists a finite covering $\xi_0 = \{\overline{V(x_1)}, \dots, \overline{V(x_n)}\}$. Consider the covering $\xi_1 = \{\overline{V(x_1)} \wedge \dots \wedge \overline{V(x_n)}\}$, which is obtained from ξ_0 by considering all possible finite intersections. Let $\xi_1 = \{\overline{U_1}, \dots, \overline{U_k}\}$. The covering ξ_1 is a partition of X_α into canonical closed sets with nonintersecting kernels, and over each $\overline{U_i}$ the bicompactum $X_{\alpha+1}$ is a direct product. Consider

$$\xi_2 = \{\overline{W_i} : \overline{W_i} = \overline{f_\alpha^{-1}(\operatorname{int} \overline{U_i})}\}.$$

By virtue of the irreducibility of $f_\alpha : D^{\tau(\alpha)} \to X_\alpha$, the covering ξ_2 is a partition of $D^{\tau(\alpha)}$ into canonical closed sets with nonintersecting kernels. Take a zero-dimensional metrizable compactum Z_α which is an irreducible preimage of the fiber K_α; let $g_\alpha : Z_\alpha \to K_\alpha$ be irreducible. For the space

$$D^{\tau(\alpha+1)} = D^{\tau(\alpha)} \times Z_\alpha$$

define the mapping $f_{\alpha+1} = f_\alpha \times g_\alpha$ by the following rule: if $y = (y_\alpha, z_\alpha) \in D^{\tau(\alpha+1)}$, with $y_\alpha \in \overline{W_i} \subset D^{\tau(\alpha)}$ and $z_\alpha \in Z_\alpha$, then put

$$f_{\alpha+1}(y) = \{\overline{f_\alpha}(y_\alpha), g_\alpha(z_\alpha)\} \in X_{\alpha+1},$$

where $\overline{f_\alpha} = f_\alpha/\overline{W_i}$. Next put $\pi_\alpha^{\alpha+1} : D^{\tau(\alpha+1)} \to D^{\tau(\alpha)}$. Since on each $\overline{W_i}$ the mapping $\overline{f_\alpha}$ is irreducible, the mapping $\overline{f_\alpha} \times g_\alpha$ is an irreducible mapping of $\overline{W_i} \times Z_\alpha$ onto $\overline{U_i}$. Further one can show that the obtained mapping $f_{\alpha+1}$ is continuous and $f_\alpha \pi_\alpha^{\alpha+1} = \varphi_\alpha^{\alpha+1} f_{\alpha+1}$. Thus, by induction, for all $\alpha < \omega(\tau)$ we construct a spectrum $\{D^{\tau(\alpha)}, \pi_\beta^\alpha\}$ and its mapping $\Phi = \{f_\alpha\}$ into the spectrum $\{X_\alpha, \varphi_\beta^\alpha\}$, which satisfies the conditions of Lemma 1. Hence we obtain that $X = \varprojlim\{X_\alpha, \varphi_\beta^\alpha\}$ is an irreducible image of

$$D^\tau = \prod_\alpha K_\alpha = \varprojlim\{D^{\tau(\alpha)}, \pi_\beta^\alpha\},$$

as was required to prove.

Theorem 1. The absolute of a factor space X of a bicompact group G is homeomorphic to pD^τ, if $wX = \tau \geq \aleph_0$.

Proof. By a theorem of E. G. Sklyarenko (4), every factor space $X = G/H$ of a bicompact group G by a closed subgroup H is representable in the form of an inverse limit

$$X = \varprojlim\{B_\alpha, \varphi_\beta^\alpha\},$$

satisfying conditions 1), 2), 3) of Lemma 2; moreover, if B_1 is discrete, then, in view of $wX \geq \aleph_0$, it suffices to put

$$B_1' = \lim_{\leftarrow}\{B_n, \varphi_n^{n+1}, n < \aleph_0\}.$$

Thus, by Lemma 2, X is an irreducible image

D^τ, whence, by the theorem of V. I. Ponomarev (^{1}), it follows that $pX = pD^\tau$, as was required to prove.

Theorem 2. *The absolute of a locally bicompact group G is homeomorphic to the product $T \times pD^\tau$, where T is a discrete space, and $\tau = \chi(e, G)$ is the local character of zero.*

Proof. By a theorem of E. G. Sklyarenko (^{4}), the space of every locally bicompact group G is homeomorphic to the topological product $T \times D^\tau \times E^n \times G'$, where E^n is Euclidean space, and G' is a connected bicompact subgroup of G. Consider the sphere S^n, which is the one-point compactification of E^n. Let $f_1 : D^{\aleph_0} \to S^n$ be an irreducible mapping of $D^{\aleph_0}$ onto S^n. Remove from $D^{\aleph_0}$ the set $f_1^{-1}(x_0)$, where x_0 is the point closing E^n to S^n. Then

$$D^{\aleph_0} \setminus f_1^{-1}(x_0) = N \times D^{\aleph_0} = Y,$$

where N is the natural row, and $f_1 : Y \to E^n$ is irreducible and perfect. Further, by Lemma 2 there exists an irreducible mapping $f_2 : D^\tau \to G'$, if $\tau = \chi(e, G')$. Thus

$$X = T \times D^\tau \times (N \times D^{\aleph_0}) \times D^\tau = T \times D^\tau$$

is irreducibly and perfectly mapped onto G. Hence $pG = T \times pD^\tau$. Theorem 2 is proved.

Theorem 3. *The absolute of the factor-space of a locally bicompact group is homeomorphic to $T \times pD^\tau$.*

§ 2. Absolutes of arbitrary topological groups

By $\Delta(X)$ we denote the dispersion character of the space X, i.e.

$$\Delta(X) = \min_{U \in X} |U|, \quad U \subset X, \quad U \text{ an open set.}$$

A space X is called topologically homogeneous if for any two points $x, y \in X$ there exists a homeomorphism $\varphi : X \overset{\text{onto}}{\to} X$ such that $\varphi(x) = y$. By G we denote an arbitrary topological group, whose group operation is written additively.

Theorem 4. *For every cardinal number $\tau \geq \aleph_0$, there exists a topologically homogeneous absolute (an extremally disconnected space) of dispersion character τ.*

Proof. Let $\pi : pG \to G$ be an irreducible perfect mapping of the absolute pG onto a topological group G for which $\Delta(G) = |G|$. Further, let $\varphi_\alpha = \varphi(x_\alpha, x) = x_\alpha + x$ be the group homeomorphism $G \to G$ carrying the zero of the group to

x_α. Then Diagram 1 is commutative, and f_α is a homeomorphism of pG onto itself. From the commutativity of Diagram 1 it follows that

$$f_\alpha(\pi^{-1}(0)) = \pi^{-1}(x_\alpha).$$

Diagram 1

Diagram 2

Diagram 3

Choose an arbitrary point $z \in \pi^{-1}(0)$. Let x_α run through G; then $z_\alpha = f_\alpha(z)$ runs through some set

$$M_z = \{z_\alpha : \; z_\alpha = f_\alpha(z), \; x_\alpha \in G\}.$$

Since $z_\alpha \in \pi^{-1}(x_\alpha)$ and the mapping π is irreducible and perfect, M_z is everywhere dense in G and, in view of the extremal disconnectedness of G, is itself extremally disconnected ([2,3]). We shall prove that M_z is topologically homogeneous. Let $z_\beta, z_\gamma \in M_z$ and $z_\beta \neq z_\gamma$. Construct a homeomorphism

$$f^\beta_\gamma : pG \to pG$$

such that

$$f^\beta_\gamma(z_\beta) = z_\gamma \quad \text{and} \quad f^\beta_\gamma(M_z) = M_z.$$

Let $z_\beta \in \pi^{-1}(x_\beta)$, and $z_\gamma \in \pi^{-1}(x_\gamma)$. Put

$$\varphi^\beta_\gamma = \varphi(x_\gamma - x_\beta, x) = x_\gamma - x_\beta + x.$$

Then Diagram 2 is commutative, with

$$\varphi_\beta = \varphi(x_\beta, x), \quad \varphi_\gamma = \varphi(x_\gamma, x), \quad \varphi^\beta_\gamma = \varphi(x_\gamma - x_\beta, x).$$

Indeed, observe that the lower base of prism 2 is commutative, since

$$\varphi^\beta_\gamma \varphi_\beta = \varphi[x_\gamma - x_\beta, \varphi(x_\beta, x)] = x_\gamma - x_\beta + x_\beta - x = x_\gamma - x = \varphi(x_\gamma, x) = \varphi_\gamma.$$

From the commu-

of the lower base and the side faces of diagram 2 we obtain the commutativity of the upper base, i.e. $f^\beta_\gamma f_\beta = f_\gamma$. Since $f_\beta(z) = z_\beta$ and $f_\gamma(z) = z_\gamma$, we have $f^\beta_\gamma(z_\beta) = z_\gamma$. We shall now prove that under the homeomorphism $f^\beta_\gamma : pG \to pG$ the set M_z is carried onto itself. Namely, let $z_1 \in M_z$ pass into $z_2 = f^\beta_\gamma(z_1)$. We shall prove that $z_2 \in M_z$. Since $z_1 \in M_z$, put $z_1 = z_\alpha = f_\alpha(z)$. Then consider $x_\xi = \varphi^\beta_\gamma(x_\alpha) = x_\gamma - x_\beta + x_\alpha$ and diagram 3, which is commutative by the commutativity of the lower base: $\varphi^\gamma_\beta \varphi_\alpha = \varphi[x_\gamma - x_\beta, \varphi(x_\alpha, x)] = x_\gamma - x_\beta + x_\alpha + x = x_\xi + x = \varphi(x_\xi, x) = \varphi_\xi$. Consequently the upper base is commutative, i.e. $f^\beta_\gamma f_\alpha = f_\xi$. Hence $f_\alpha(z) = z_\alpha = z_1$ and $f^\beta_\gamma(z_1) = f_\xi(z) = z_\xi \in M_z$. Thus $z_\xi = z_2$, and the topological homogeneity of M_z is completely proved. We shall prove that $\Delta(M_z) = \tau$. Let U be an arbitrary open set in M_z. Then $U = M_z \cap V$, where V is open in pG. Since the map π is irreducible and perfect, there exists a nonempty open set $W \subset G$ such that $\pi^{-1}(W) \subset V$. Further, M_z is dense in pG; consequently, U is dense in V, and hence $U' = M_z \cap \pi^{-1}(W) \subset M_z \cap V = U$. By the very construction of the set M_z, if $x \in W$, then $\pi^{-1}(x) \cap M_z \neq \emptyset$, therefore $|U| \geq |\operatorname{int} \pi(U)| \geq \Delta(G)$. Thus we have proved that for every open $U \subset M_z$ we have $\operatorname{int} \pi(U) \neq \emptyset$. Hence $|U| \geq |\operatorname{int} \pi(U)| \geq \Delta(G)$. Consequently, $\Delta(M_z) \geq \Delta(G)$. On the other hand, since π is one-to-one on M_z, $\Delta(M_z) \leq |M_z| = |G| = \Delta(G)$. Finally, as G take the subgroup σ of the generalized Cantor discontinuum D^τ, $\tau \geq \aleph_0$, consisting of points $\{x_\alpha\} \in D^\tau$ for which only a finite number of coordinates are different from zero. It can be shown that $\Delta(\sigma) = |\sigma| = \tau$. Denote by $M_z(\sigma)$ the set M_z constructed for the group σ. By what was proved above, $M_z(\sigma)$ is a topologically homogeneous, extremely disconnected space, and $\Delta(M_z(\sigma)) = \Delta\sigma = \tau$. Theorem 4 is completely proved.

Theorem 5. *For every cardinal number $\tau \geq \aleph_0$ there exists a space X possessing the following properties: 1) X is extremely disconnected; 2) X is topologically homogeneous; 3) X is hereditarily normal; 4) X is hereditarily paracompact; 5) X is non-discrete; 6) $|X| = \tau$.*

For the proof of this theorem it suffices to consider the product $X = M_z \times T$, where T is a discrete space of cardinality τ and M_z is the space constructed in Theorem 4, if the group G is the group of rational numbers.

It can be shown that the weight of the absolute pC of the Cantor perfect set C is equal to the continuum, whence, by a theorem of A. V. Arhangel' skii (^{6}), pC is not topologically homogeneous. Nevertheless pC decomposes into topologically homogeneous absolutes. This follows from the following theorem.

Theorem 6. *The absolute pG of any topological group G decomposes into everywhere dense disjoint topologically homogeneous absolutes, namely,*

$$pG = \bigcup_{z \in \pi^{-1}(0)} M_z.$$

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Note: Figure translations are in progress. See original paper for figures.

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