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Abstract

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NORMAL DIVISORS OF A FINITE GROUP

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1. In group theory it is important not only to establish the existence of a normal divisor, but also its structure and position in the group. In this note a number of results related to this circle of questions are formulated. Only finite groups are considered.

In §2 the discussion concerns p-groups. Theorem 1 gives a description of a p-group G with cyclic Frattini subgroup $\Phi(G)$. It is easy to see that this class includes p-groups in which all abelian characteristic subgroups are cyclic, so that F. Hall' s theorem on such groups (a shorter, more elementary proof of it was published by D. Gorenstein ([1])) is a special case of Theorem 1. We note that the class of groups considered in Theorem 1 is hereditary for subgroups and homomorphic images, which cannot be said of the class of groups described by F. Hall' s theorem. A corollary of Theorem 2 refines a theorem of C. Hobby ([2]). Theorem 3 contains the formulation of a series of new theorems on the number of subgroups of a given order and a given structure in a nonmetacyclic p-group, $p > 3$. Known theorems of G. Miller and M. Tazawa are special cases of Theorem 3. From this theorem two corollaries on normal divisors in a p-group are derived (these corollaries in form resemble N. Blackburn' s well-known theorem on p-groups without normal divisors of type (p, p, p)). We note that for regular groups Theorem 3 is true without restrictions on p. Theorem 4 gives examples of 2-groups G for which $\mathrm{Aut}(G)$ are 2-groups.

In §3 several results are formulated on groups representable as a product of pairwise permutable subgroups. In Theorem 11, Γ_1^p-quasinilpotent groups are studied; a complete description is given of solvable groups with this property. In particular, it turns out that Γ_1^2-quasinilpotent groups are always solvable, while the question of solvability of Γ_1^p-quasinilpotent groups, if p is odd, remains open.

Notation and definitions. H^G is the normal closure of the subgroup H in the group G; π is the set of prime numbers; Γ_1 is the set of all maximal subgroups of the group G; Γ_1^p is the set of all those elements of Γ_1 whose order is divisible by p (in the case when the set Γ_1^p is under discussion, we always assume that the order of the group is divisible by p); $N(H)$ is the normalizer, and $C(H)$ is the centralizer of the complex H in G; two subgroups F and H form a nilpotent pair if they are either incident, or

$$N(F \cap H) \cap F \neq F \cap H \neq N(F \cap H) \cap H;$$

if M is a set of subgroups in G, then we shall call the latter M-quasinilpotent if any two elements of M form a nilpotent pair; a p-group is called extraspecial if its center, commutator subgroup, and Frattini subgroup coincide and have order p (for convenience, the identity group is also counted as extraspecial); $Z(G)$ is the center of the group G; G' is the commutator subgroup of the group G.

2. Theorem 1. *Let the Frattini subgroup of a nonabelian p-group G be cyclic, and let Φ_0 be a subgroup of order p in $\Phi(G)$. Then $G = AECM$, where*

in addition: (a) the subgroup A is characteristic in G, abelian, and has exponent p; (b) the subgroup E is extraspecial, and, if p is odd, the exponent of E does not exceed p; (c) $C = Z(EC)$ is cyclic; (d) the subgroup AEC is generated by Φ_0 and all subgroups containing Φ_0 and invariant in G of type (p, p) (in particular, AEC/Φ_0 has exponent not exceeding p, and lies in $Z(G/\Phi_0)$); (e) the subgroup M is cyclic or a 2-group of maximal class, and $\Phi(G) \subseteq M$; (f) if M is cyclic, then $\Phi(G) \subseteq Z(G)$ and G' has order p; (g) if M is of maximal class, then $G' = \Phi(G)$ and $|G : C(\Phi(G))| = 2,\ |M| = 4|\Phi(G)|$; (h) if A is cyclic, then $C \cdot \Phi(G)$ is an abelian characteristic subgroup of G.

In particular, if the p-group G contains no noncyclic characteristic abelian subgroups, then $\Phi(G)$ is cyclic, $A \subseteq \Phi_0,\ C \subseteq M$, and $G = EM$; this is precisely the theorem of P. Hall ([1]).

Theorem 2. Let a p-group G be contained as an invariant subgroup in an arbitrary group H; let $\Phi_H(G)$ be the intersection of all H-admissible maximal subgroups in G; and let an H-admissible subgroup N be supersolubly embedded in H and lie in $\Phi_H(G)$. If N is generated by two elements, then it is metacyclic, provided $N - \Phi(G)$ contains no elements of order 4 (the intersection of the empty set of subgroups of G is taken to be G).

Corollary 1. Let a subgroup N be generated by two elements and be invariant in the p-group G. If $N \subseteq \Phi(G)$, then N is metacyclic.

Theorem 3. Let G be a nonmetacyclic p-group, $p > 3$. Then the following assertions hold:

(a) The number of subgroups of order p^n, $n > 2$, contained in G and metacyclic is divisible by p.

(b) The number of cyclic subgroups of order $p^n,\ n > 1$, contained in G, is divisible by p^2.

(c) G contains a number divisible by p of such subgroups of order $p^n,\ n > 2$, each of which possesses a cyclic subgroup of index p^2.

(d) The number of subgroups of order p contained in G is congruent to $1+p+p^2$ modulo p^3.

Corollary 2. If $p > 3$, then an invariant subgroup M in a p-group G is metacyclic if and only if it contains no subgroup invariant in G of order p^3 and exponent p.

Corollary 3. If $p > 3$, then the Frattini subgroup of a p-group G is metacyclic if and only if $\Phi(G)$ contains no subgroup invariant in G of type (p, p, p).

Theorem 4. Let $P_i = G_i \times C$, where the subgroup C is a cyclic 2-group, G_1 is a 2-group of maximal class, and

$$G_2 = \mathrm{gp}\langle a, b \mid a^2 = b^{2^{n-1}} = 1,\ aba = b^{1+2^{n-2}},\ n > 3\rangle.$$

If G_1 is not a quaternion group, then $\mathrm{Aut}(P_i)$ is a 2-group. Moreover, if P_2 has odd index in a finite group G, then G has an invariant 2-complement.

Theorem 5. Let a group G have odd order p^{3m}, $m > 3$. Then the number of nonabelian subgroups of order p^3 contained in G is divisible by p.

For 2-groups, a stronger result has been proved in ([4]) (if G is not a group of maximal class, then the number indicated in the theorem is even divisible by 4).

In the proof of Theorem 4, Frobenius' theorem on an invariant p-complement is used essentially, together with the theorem of Grün–Hall.

3. **Theorem 6.** Let a soluble group $G = A \cdot B_1 \cdot \ldots \cdot B_n$, where the subgroups B_i are nilpotent and A is a p-closed group for all p dividing $|B_1 \cdot \ldots \cdot B_n|$. If $A \neq G$, then the normal closure of one of the subgroups $A, B_1, \ldots, B_n$ is distinct from G.

This theorem generalizes Kegel' s well-known result on normal subgroups of sets in the product of two nilpotent groups.

A Z_π-group is a group with cyclic Sylow p-subgroups for all p in π. A group is called π-decomposable if its nilpotent π-Hall subgroup is a direct factor.

Theorem 7. *Let $|G| = mn$, where m is the greatest π-divisor of $|G|$, and for all p in π we have $(p-1, n) = 1$. If G is the product of two π-decomposable Z_π-groups, then it contains an invariant subgroup of order n.*

The proof of this theorem is based on the use of results of V. D. Mazurov ([3]) and of the author ([4]) on automorphisms of the product of two cyclic 2-groups.

Theorem 8. *Let $G = (P_1 \times L_1) \cdot (P_2 \times L_2)$, where P_i is a Sylow 2-subgroup in $P_i \times L_i$. If the P_i are metacyclic, then either G is solvable, or G contains a subgroup whose homomorphic image is $PSL(2, p)$, where p is a Mersenne prime.*

The core H_G of a subgroup H in a group G is the intersection of all subgroups conjugate to H in G. For the definition of a $\pi\varphi$-dispersive group, see ([5]).

Theorem 9. *Let all maximal subgroups of a group G that have a given core H_G be $\pi\varphi$-dispersive. Then either G is a π-solvable group with π-length not* *exceeding 2, or 2 is contained in π, and the (unique) non-π-solvable chief factor group of G contains a subgroup whose homomorphic image is the symmetric group of degree four.*

Theorems 10 and 11 were proved jointly by S. L. Gramm and the author.

Theorem 10. *Let the set of prime divisors of the index of a maximal non-nilpotent solvable subgroup H in G be independent of H (it is, of course, assumed that G contains at least one non-nilpotent solvable subgroup). Then one of the following assertions holds:*

(a) $G = P \times L$, where P is a Sylow subgroup in G, and L is a Schmidt group.

(b) $|G|$ contains exactly two distinct prime numbers, and in the factor group of G modulo the hypercenter one of the Sylow subgroups is maximal.

Theorem 11. *Let G be a Γ_1^p-quasinilpotent group of order divisible by p. Then:*

(a) *If G is solvable, then: (1a) any two elements of Γ_1^p are conjugate in G; (2a) the order of the intersection of any two distinct elements of Γ_1^p is not divisible by p; (3a) $G/\Phi(G)$ is an elementary group generated by two elements of order p.*

(b) *If G is solvable, then it is either Γ_1-quasinilpotent, or is an extension of a minimal invariant p-subgroup by means of a Γ_1-quasinilpotent group of order not divisible by p.*

In particular, from part (a) of the theorem it follows that Γ_1^2-quasinilpotent groups are solvable, while the question of solvability of Γ_1^p-quasinilpotent groups for $p > 2$ remains open. The theorem also easily yields the solvability of a π-separable Γ_1^p-quasinilpotent group in the case where $|G|$ is divisible by p and at least two prime numbers from π divide $|G|$.

Theorem 12. *A group G is π-solvable if and only if it satisfies the following conditions:*

(a) *If M is an arbitrary normal divisor in G, then in it there is a π-Hall subgroup P such that $M \cdot N(P) = G$.*

(b) *If H is maximal in G and its index in G is a π-number, then for all maximal subgroups in G having core H_G, there is a prime number dividing their indices in G.*

From this theorem one obtains the following necessary and sufficient condition for the conjugacy of maximal subgroups of a π-solvable group:

Let F and H be maximal subgroups of a π-solvable group G, and suppose that the index of F in G is a π-number. Then F and H are conjugate in G if and only if their Sylow subgroups corresponding to the same prime numbers are conjugate in G.

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Note: Figure translations are in progress. See original paper for figures.