ON THE EXTENSION OF A NORMAL TOPOLOGY IN A \(K\)-LINEAL

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Abstract

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ON THE EXTENSION OF A NORMAL TOPOLOGY IN A K-LINEAL

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1. A fundamental system of neighborhoods of zero of a linear topological space X is called a basis. If the space X is Hausdorff, is a K-lineal, and has a basis consisting of normal (1) neighborhoods, then we shall agree to call the topology and the basis **normal**, and X a KT-lineal. Convergence of a net $x_\alpha \to x$ is understood to be topological. A KT-lineal has the basic properties of a KN-lineal (1), following from the normality of convergence: if $|x_\alpha| \leq |y_\alpha|$, $y_\alpha \to 0$, then $x_\alpha \to 0$. A (topologically) countably complete KT-lineal with unit (b) is complete in the sense of (2–4).

2. Let a KT-lineal X with a normal basis $\{V\}$ be a linear substructure of a K-lineal Y (in particular, when Y is its K-completion) such that for every element $y > 0$ there exist elements $x_i: \ 0 < x_1 \leq y \leq x_2$. Then the normal topology from X extends to Y. Put $y \in W$ if in the neighborhood V there exists an element $x \geq |y|$. In this case Y becomes a KT-lineal with normal basis $\{W\}$; $V = W \cap X$.

3. If in some K-lineal X, which is a Hausdorff linear topological space, convergence of a sequence is normal (or X is Archimedean, and from (r)-convergence of a sequence to zero there follows convergence to zero), then X can be turned into a KT-lineal with a stronger topology. If, however, convergence of a net in X is normal, then X is turned into a KT-lineal with the same topology. A normal basis $\{V\}$ is constructed on the basis of the given basis $\{B\}$ of the space X: we set $y \in V$ if from the relation $|x| \leq |y|$ it always follows that $x \in B$.

4. In a similar way one can extend a normal topology from a certain subset to the whole K-lineal. Let Z be an Archimedean K-lineal, and suppose it contains a normal subset S of certain elements $s \geq 0$, structurally complete in Z, i.e., for any $z \neq 0$ there exists an s such that $|z| \wedge s > 0$. Further, suppose there is a system of subsets $B \subset S$ satisfying the requirements:

$(B\ 1)$ For any B_i there exists $B \subset B_1 \cap B_2$.

$(B\ 2)$ If $s_n \xrightarrow{(r)} 0$, then for every B there exists an n such that $s_n \in B$.

(B 3) For every B_0 there exists B such that $B + B \subset B_0$.

(B 4) $\bigcap B = \{0\}$.

Define neighborhoods V_B of zero in Z by putting $z \in V_B$ if from the comparability $|z| \geq s$ it always follows that $s \in B$. In this case Z becomes a KT-lineal with normal basis $\{V_B\}$. $S \cap V_B \subset B$. If the B' s are normal, then

$$S \cap V_B = B.$$

Thus, the normal topology from S is extended to Z.

Let us verify the fulfillment of the conditions of the converse theorem 1(1.XI) from (^{5}) and Hausdorffness (without requiring the normality of B). If B is chosen according to condition (B 1), then $V_B \subset \bigcap_i V_{B_i}$. The assertion that the element z is not absorbed by the set $\hat{V}_B$ is equivalent to the assertion of the existence of elements $s_n \in B$, $|z| \geq ns_n$ for all n, which contradicts condition (B 2). If B satisfies condition (B 3), then $V_B + V_B \subset V_{B_0}$. Indeed, let

$z_i \in V_B$, $|z_1 + z_2| \geq s$. In the K-lineal Z, the element s decomposes into a sum of elements s_i, $0 \leq s_i \leq |z_i|$. Since $s_i \leq s$ and S is normal, $s_i \in S$. From the relations $z_i \in V_B$, $|z_i| \geq s_i$ it follows that $s_i \in B$, $s \in B + B \subset B_0$. Thus, from the comparability $|z_1 + z_2| \geq s$ it follows that $s \in B_0$, and therefore $z_1 + z_2 \in V_{B_0}$.

Let us establish separatedness. Let $z \in \bigcap V_B$, $|z| \geq s$; then $s \in \bigcap B$, $s = 0$ by condition (B 4). By virtue of normality and the structural completeness of S in Z, it follows that $z = 0$.

5. Let some KT-lineal Z_0 with normal basis $\{W\}$ be an Archimedean foundation of the K-lineal Z. Put

$$S = (Z_0)_+, \qquad B = W \cap S.$$

Then B is normal, satisfies conditions (B 1–4), and $W \subset V_B$; Z_0 is (topologically) homomorphically embedded in Z.

6. An Archimedean (b)-complete in the sense of $^{(2-4)}$ K-lineal with unit (a.p.e. K-lineal) is realized $^{(3,4)}$ by the K-lineal $Z = Z(Q)$ of certain continuous functions, admitting infinite values, on some bicompact $^{(1)}$ Q. The K-lineal $Z_0 = C(Q)$ of all continuous bounded functions $^{(2)}$ is contained in Z. The basis I of unit elements e is structurally isomorphic to the system of open-closed sets $E_e \subset Q$. In the K-lineal Z, when there is an arbitrary normal topology of the basis, separated at zero, an analogue of the topology of the space of measurable functions can be constructed by the method given in $^{(6)}$ for a universal semiring, i.e., an extended K-space.

We shall prove this fact under weaker restrictions, as compared with $^{(6)}$, on the K-lineal $Z(Q)$ and the topology of its basis I.

Let there exist a system of subsets $\Gamma \subset I$ satisfying the conditions:

(Γ 1) For any Γ_i there exists $\Gamma \subset \Gamma_1 \cap \Gamma_2$.

(Γ 2) For each Γ_0 there is a Γ such that, if $e_i \in \Gamma$, then $e_1 \vee e_2 \in \Gamma_0$.

(Γ 3) Γ is normally contained in I.

(Γ 4) $\bigcap \Gamma = \{0\}$.

Define the sets $W_{\Gamma\varepsilon}$ ⁽⁶⁾: $z \in W_{\Gamma\varepsilon}$ if there exists an element $e \in \Gamma$ such that

$$\{t;\ |z(t)| \geq \varepsilon\} \subset E_e,$$

where $0 < \varepsilon \leq 1$.

Then the K-lineal Z_0 of bounded elements becomes a KT-lineal with normal basis $\{W_{\Gamma\varepsilon}\}$. If, however, the bicompact Q is totally disconnected (1) (a particular case of such a K-lineal $Z(Q)$ is a K_σ-space (1) with unit), then all of Z is a KT-lineal with the same basis. The system $\{\Gamma\}$ defines a normal regular topology of the basis I, extended to Z_0 or Z,

$$\Gamma = I \cap W_{\Gamma\varepsilon}. \tag{1}$$

We shall prove this assertion. If $0 < \varepsilon \leq \varepsilon_i$, $\Gamma \subset \bigcap_i \Gamma_i$ (condition (Γ 1)), then $W_{\Gamma\varepsilon} \subset \bigcap_i W_{\Gamma_i\varepsilon_i}$. If Γ is taken according to condition (Γ 2), then $W_{\Gamma\varepsilon} + W_{\Gamma\varepsilon} \subset W_{\Gamma_0,2\varepsilon}$.

The bounded elements are absorbed by any set $W_{\Gamma\varepsilon}$. Further, let $z \in \bigcap W_{\Gamma\varepsilon}$. Fix ε. By the definition of $W_{\Gamma\varepsilon}$ there exist elements $e_\gamma \in \Gamma$ such that the function $|z(t)| < \varepsilon$ on all $E_{Ce_\gamma} = Q \setminus E_{e_\gamma}$. Let $e \leq e_\gamma$ for all γ. By condition (Γ 3), $e \in \Gamma$, and by condition (Γ 4), $e = 0$. Consequently, $\inf e_\gamma = 0$, i.e. $\bigcup E_{Ce_\gamma}$ is dense in Q. But on this union $|z(t)| < \varepsilon$, and ε is arbitrary; hence $z = 0$. Thus Z_0 is a KT-lineal with normal basis $\{W_{\Gamma\varepsilon}\}$.

Next, take

$$S = (Z_0)_+, \qquad B = S \cap W_{\Gamma\varepsilon}.$$

Then (see above, item 5) Z is a KT-lineal with basis $\{V_B\}$ and

$$W_{\Gamma\varepsilon} \subset V_B. \tag{2}$$

Now let Q be totally disconnected. We shall prove that

$$\mu V_B \subset W_{\Gamma\varepsilon} \quad \text{for } 0 < \mu < \varepsilon. \tag{3}$$

If $|z(t_0)| > \mu$, then there exists an open-and-closed neighborhood E_e of the point t_0 in the totally disconnected bicompact Q such that $|z(t)| > \mu$ on E_e, i.e. $|z| > \mu e$. Therefore

$$\{t;\ |z(t)| > \mu\} \subset \bigcup_{|z| > \mu e} E_e.$$

The closed set $\{t;\ |z(t)| \geq \varepsilon\}$, for $\mu < \varepsilon$, is covered in the bicompact Q by a finite collection of sets E_{e_i}, where $|z| > \mu e_i$. For $e = \sup e_i$,

$$\{t;\ |z(t)| \geq \varepsilon\} \subset E_e, \qquad |z| \geq \mu e.$$

If $z \in \mu V_B$, then from the relations $|z| \geq \mu e$, $\mu > 0$, $e \in S$, it follows that $e \in B \cap I = \Gamma$, by relation (1). Hence $z \in W_{\Gamma e}$, and inclusion (3) is established. But from inclusions (2), (3) it follows that the system $\{W_{\Gamma e}\} \sim \{V_B\}$, taken as a base, turns Z into a KT-lineal. If $e_\alpha \in I$, $e_\alpha \to z$ in Z_0 or Z, then

$$0 = e_\alpha \wedge (1 - e_\alpha) \to z \wedge (1 - z), \qquad z \in I.$$

The base I is a (topological) subspace of the KT-lineal.

We note that the requirements postulated by axiom 6 on p. 52 in (6) are not imposed here.

The topology determined by the base $\{W_{\Gamma e}\}$ is an analogue of the topology of a space of measurable functions; the sets E_e play the role of measurable sets in the realization of KB-spaces (7-10).

7. Let the conditions of item 5 be satisfied, with the notation Z_0 replaced by X, and W by U. Put $\Gamma = U \cap I$; we find that conditions (Γ 1–4) are satisfied. If Q is totally disconnected, then

   $$\mu U \subset W_{\Gamma e} \qquad (0 < \mu < \varepsilon),$$

   and X is embedded homeomorphically in Z.

8. In the KT-lineal Z (or Z_0) with base $\{W_{\Gamma e}\}$, the abstract function generated by (4) from a real continuous function of several variables is topologically continuous at all points in whose neighborhood the abstract function is meaningful.

9. If a KT-lineal X with unit and normal base $\{U\}$ is a normal sublineal (1) of some internally normal (4) a.l. K-lineal Z with the same unit (a particular case of such a K-lineal Z is a K_σ-space with unit), the neighborhoods Γ are defined by equality (4), and the requirement stated in axiom 8 ((6), p. 52) is fulfilled, then the mapping (see above, item 7) of X onto its image is a (topological) homeomorphism.

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Note: Figure translations are in progress. See original paper for figures.

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