LOCAL CONTRACTIBILITY OF THE GROUP OF HOMEOMORPHISMS OF A MANIFOLD

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Abstract

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LOCAL CONTRACTIBILITY OF THE GROUP OF HOMEOMORPHISMS OF A MANIFOLD

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1. Statement of the problem. In this note an outline is given of a proof of the local contractibility of the group of homeomorphisms $\mathfrak{H}$ of a metrizable manifold M (with metric $\rho(x, y)$), endowed with one of three topologies—compact-open, uniform, or majorant. A base of neighborhoods of the identity e is given in the first case by pairs (K, ε): the neighborhood $\Omega_{K,\varepsilon}(e)$, where K is compact and $\varepsilon > 0$, consists of homeomorphisms h for which $\rho(x, hx) < \varepsilon$ for $x \in K$; in the second case by numbers $\varepsilon > 0$: the neighborhood $\Omega_\varepsilon(e)$, determined by the number $\varepsilon > 0$, consists of homeomorphisms h for which $\rho(x, hx) < \varepsilon$ for all x; in the third case by strictly positive functions f, which we shall call majorants: the neighborhood $\Omega_f(e)$, determined by the majorant $f : M \to (0, \infty)$, consists of homeomorphisms for which $\rho(x, hx) < fx$ for all x. The correspondingly topologized group is denoted by $\mathfrak{H}_c(M)$, $\mathfrak{H}_u(M)$, or $\mathfrak{H}_m(M)$, or by $\mathfrak{H}_\tau(M)$, if it is not specified which topology is meant. For a compact manifold all three topologies coincide.

An isotopy of the manifold M is a fiber-preserving homeomorphism of $M \times I$ onto itself. The condition of being fiber-preserving means that the isotopy Φ determines homeomorphisms $(\Phi)_t$ so that $\Phi(x, t) = ((\Phi)_t x, t)$. We topologize the set of all isotopies $\mathfrak{I}_\tau(M)$ as a subspace of $\mathfrak{H}_\tau(M \times I)$. The identity in $\mathfrak{I}_\tau$ will be denoted by E. Besides the product $\Phi\Psi$ of isotopies Ψ and Φ in the sense of this group, we also consider the composition $\Phi \circ \Psi$, which is defined as for homotopies if $(\Phi)_0 = (\Psi)_1$.

We say that a subset $A \subset \mathfrak{H}_\tau$ is deformed along a subset B into a subset Γ, if there exists a continuous mapping $\Upsilon : A \to \mathfrak{I}_\tau$ such that for $h \in A$ we have $(\Upsilon(h))_0 = h$, $(\Upsilon(h))_t \in B$, $t \in I$, $(\Upsilon(h))_1 \in \Gamma$. In particular, if A is a neighborhood of e, and $\Gamma = e$, then we shall say that the group $\mathfrak{H}_\tau$ is locally contractible. By $\Delta_\tau(X)$ we shall denote the subgroup of $\mathfrak{H}_\tau(M)$ consisting of homeomorphisms identical on X.

Main theorem. *For an arbitrary metrizable manifold M, the group $\mathfrak{H}_\tau(M)$ is locally contractible.*

Since for compact M all topologies coincide, we have the following consequence:

Theorem 1. *For a compact manifold M, the group $\mathfrak{H}_c(M)$ is locally contractible.*

In the case of noncompact M, as simple examples show, one can speak of local contractibility of $\mathfrak{H}_c(M)$ or $\mathfrak{H}_u(M)$ only when the topology of M is sufficiently simple at infinity. At the same time, for an open manifold with finitely generated homology and one-connected at infinity in the case when $\dim M \geq 6$, or when M is three-dimensional and irreducible ([1,2]), there exists a compact manifold whose interior is homeomorphic to M. Therefore, although in general we cannot give an answer for the topologies $\tau = c$ and u, with a high degree of generality one may assume that if M is open, then it is the interior of a compact-

of the manifold N. In this case, using Brown' s result (3), that ∂N has in N a neighborhood G of the form $\partial N \times I$, and Alexander' s modification of the argument (4), we can prove that the subgroup $\Delta_\tau(M \setminus G)$ is contractible, where $\tau = c$ or u, and it is assumed that the metric of M is induced by the metric of N. Thus, it remains only to prove that $\mathfrak{H}_\tau(M)$ is locally contractible in $\Delta_\tau(K)$. This follows easily from the following more general proposition, from which, as a special case, the main theorem is also obtained.

(A). *If C and D are closed subsets of the manifold M, then for every neighborhood $O(C)$ and every neighborhood $O'(\operatorname{Fr} D \cap C)$ there exists a strictly positive function f on O such that for every homeomorphism h, for which $\rho(x, hx) < fx,\ x \in O$, there is an isotopy $\Upsilon(h)$, continuously depending on h, such that: 1) $(\Upsilon(h))_0 = h$; 2) $(\Upsilon(h))_t = h$ on $M \setminus O,\ t \in I$; 3) $(\Upsilon(h))_t = e$ on $D \setminus O',\ t \in I$; 4) $(\Upsilon(h))_1 = e$ on C.*

If in the case indicated above, where $M = \operatorname{Int} N$, we take $[M \setminus G]$ as C and $D = \Lambda$, then, as was said, we obtain the following consequence:

Theorem 2. *If $M = \operatorname{Int} N$, where N is a compact manifold, then $\mathfrak{H}_c(M)$ and also $\mathfrak{H}_u(M)$, if the metric of M is induced by the metric of N, are locally contractible.*

2. Reduction of (A) to the main lemma. Without much difficulty it is shown that it is enough to prove (A) for manifolds without boundary and, moreover, in the case when C is compact. We shall also assume that the closure of O is compact. Choose a covering of C by Euclidean neighborhoods:

$$C \subset \bigcup_{i=1}^{k} Q_i, \quad \text{where } Q_i = q_{iR}^n, \quad \text{and } q_i : R^n \to M \text{ is a homeomorphism}, \quad 1 \leq i \leq k.$$

Let I_r^n be the cube in R^n: $\{x \mid |x_i| \leq r,\ 1 \leq i \leq n\}$, and let $I^n = I_1^n$. We shall assume that $C \subset \bigcup q_i I^n$, and moreover $\operatorname{Fr} D \cap \bigcup [Q_i] \subset O$, $\operatorname{Fr} D \cap \bigcup [Q_i] \subset O'$. Take another k neighborhoods $O_i(\operatorname{Fr} D \cap C)$ such that $[O_i] \subset O_{i+1} \subset O'$, and one may assume that $\operatorname{Fr} D \cap [Q_i] \subset O_1$. Take also numbers $\gamma_i,\ 1 \leq i \leq k$, such that $0 < \gamma_i < \gamma_{i-1} \leq \gamma_1 < 1/4$. We construct successively numbers δ_i and, for every homeomorphism

$$h \in \Omega_{[O],\delta_i}(e) \cap \Delta_\tau(D),$$

isotopies $\Psi_i(h)$, continuously depending on h, such that: 1) $(\Psi_i(h))_0 = (\Psi_{i-1}(h))_1$ (or $= h$ for $i = 1$); 2) $(\Psi_i(h))_t = (\Psi_{i-1}(h))_1$ on $M \setminus Q_i$ (or $= h$ for $i = 1$); 3) $(\Psi_i(h))_1 = e$ on $D \setminus O_i$; 4) $(\Psi_{i-1}(h))_1 = e$ on

$$\bigcup_{i'=1}^{i-1} q_{i'} I^n_{1+\gamma_i}.$$

If

$$\Upsilon(h) = \Psi_k(h) \circ \dots \circ \Psi_1(h)$$

for a homeomorphism h from $\Omega_{[O],\delta}(e) \cap \Delta(D)$, where $\delta > 0$ is sufficiently small, then it is easy to show that $\Upsilon(h)$ has the required properties. Condition 2 shows that the construction of Ψ_i takes place in a single neighborhood Q_i and, consequently, can be transferred to Euclidean space. Namely, it is enough to prove the following assertion:

(B). *If D is a closed subset of R^n, then for every neighborhood $O(\operatorname{Fr} D \cap I^n_2)$ there exists $\delta > 0$ such that for every δ-homeomorphism $g : I^n_2 \to R^n$, identical on D, there is an isotopy $\Psi(g)$ of the space R^n, continuously depending on g, such that: 1) $(\Psi(g))_0 = e$; 2) $(\Psi(g))_t = e$ on $(R^n \setminus gI^n_2) \cup (D \setminus O)$; 3) $(\Psi(g))_1 = g^{-1}$ on $I^n_{1.5}$.*

To prove (B) we take a triangulation T of R^n so fine that

$$\operatorname{St}_{T''}(\operatorname{St}_T(D \setminus O) \cap (\operatorname{Fr} D \cap I^n_2)) = \Lambda$$

and

$$\operatorname{St}_{T'} \operatorname{St}_T I^n_{1.5} \subset I^n_{1/4},$$

where T'' is the second barycentric subdivision. For each open simplex $\sigma \in T$ take the cell

$$z = z(\sigma) = [\operatorname{St}_{T''} \sigma \setminus \operatorname{St}_{T''} \partial\sigma].$$

Let $\sigma_1, \dots, \sigma_s$ be the simplexes from $\operatorname{St}_T I^n_{1.5}$, where first the simplexes from $[\operatorname{St}_T(D \setminus O)]$ are numbered (say, d of them), and the remaining $s - d$ in order of increasing dimension. It is easy to see that if $z_i \cap (D \setminus O) \neq \Lambda$, then $i \leq d$. Let the numbers $\eta_i > 0$, $d + 1 \leq i \leq s$, be so small that $O_{\eta_i}(z_i) \subset I^n_2$, $O_{\eta_i}(z_i) \cap O_{\eta_i}(z_{i'}) = \Lambda$, if $z_i \cap z_{i'} = \Lambda$, and if $O_{\eta_i}(z_i) \cap (D \setminus O) \neq \Lambda$, then $i \leq d$. We construct successively numbers δ_i and, for every δ_i-homeomorphism identical on D, an isotopy Φ_i, continuously depending on g, such that the first d of them are ident-

are the identity, and $(\Phi_i(g))_0 = (\Phi_{i-1}(g))_1$, $(\Phi_i(g))_t = (\Phi_{i-1}(g))_1$ on $(R^n \setminus gO^i_\eta(z_i)) \cup \operatorname{St}_{r'} \partial\sigma_i$, $(\Phi_i(g))_1 = g^{-1}$ on gz_i. Then, for a δ-homeomorphism $g : I^n_2 \to R^n$, where δ is sufficiently small, all the Φ_i are defined and, hence, so is their composition Ψ, which, as is easy to see, has the required properties. To construct the next isotopy Φ_i we construct a homeomorphism $q : R^n \to O_{\eta_i}$ such that

$$qI^n = z_i, \qquad q(R^n \setminus (I^p \times R^{n-p})) \subset \operatorname{St}_{r'} \partial\sigma_i,$$

$$q(I^p \times \bar{R}^{n-p}) \subset O_\eta(z_i),$$

where $p = \dim \sigma_i$ and $R^n = R^p \times R^{n-p}$. It is easy to show that, with the aid of the homeomorphism q, the construction of Φ_i reduces to the following lemma.

3. Main lemma. *There exists a $\delta > 0$ such that for every homeomorphic δ-shift $g : I_2^n \to R^n$, identical on $I_2^n \setminus I^p \times I$, there exists an isotopy $\mathrm{H}(g)$, depending continuously on ξ, such that:*

1) $(\mathrm{H}(g))_0 = e$;

2) $(\mathrm{H}(g))_t = e$ on $R^n \setminus g(I^p \times I_2^{n-p})$, $t \in I$;

3) $(\mathrm{H}(g))_1 = g^{-1}$ on gI^n.

The general case is easily obtained from the case $p = 0$, which we shall assume in what follows. We put $\delta = 1/8 \cdot 3^{2n}$. Introduce the notation:

$$R_{i,d} = \{x \mid x_i = d\},$$

$$R_{i,d}^+ = \{x \mid x_i \geq d\}, \qquad R_{i,d}^- = \{x \mid x_i \leq d\}, \qquad R_i(d_1, d_2) = R_{i,d_1}^+ \cap R_{i,d_2}^-,$$

$$R_{i,d}^\varepsilon = R_i(d - \varepsilon, d + \varepsilon), \qquad I_i(d, r) = R_{i,d} \cap I_r^n,$$

$$I_i^\varepsilon(d; r) = R_{i,d}^\varepsilon \cap I_r^n, \qquad \Pi_i(d_1, d_2; r) = R_i(d_1, d_2) \cap I_r^n.$$

By $\xi_i(d_1, d_2, d_3, d_4; r_1, r_2)$, where $d_1 < d_2$, $d_3 < d_4$, we denote an isotopy which is the identity outside $\Pi_i(d_1, d_2; r_2)$, on each segment l parallel to Ox_i and contained between $I_i(d_1; r_1)$ and $I_i(d_4; r_1)$, carries the point $l \cap I_i(d_1; r_1)$ to the point $l \cap I_i(d_3; r_1)$, and is extended linearly to the adjacent intervals of this segment and to the segment of each ray issuing from the point $R_{i,d} \cap Ox_i$, contained between $I_{r_2}^n$ and $I_{r_1}^n$. It is clear that $\xi_i(d_1, d_2, d_3, d_4; r_1, r_2)$ carries into $I_i(d_2; r_1) = I_i(d_3; r_1)$.

We shall construct a sequence of isotopies Φ_j, depending continuously on g, so that $(\Phi_j)_0 = e$ and, if $g_j = (\Phi_j)_1 \dots (\Phi_1)_1 g$ (and $g_0 = g$), then we can define

$$(\mathrm{H}(g))_t = (\Phi_j)_{(j+1)(tj+1-j)} g_{j-1}, \qquad t \in [j/(j+1), (j+1)/(j+2)],$$

and $(\mathrm{H}(g))_1 = \lim g_j$. It is easy to verify that for this it is sufficient that, first, each Φ_j be an $(\eta/2^j)$-isotopy, where η does not depend on j or on g, and, second, that

$$g_j I_i(d; r) \subset \operatorname{Int} I_i^{\varepsilon_j}(d; r + \varepsilon_j),$$

where $1 \leq i \leq n$, $r = 1 + 1/2^j$, $\varepsilon_j = 1/2^{j+1} 3^{2n}$, and d runs through all rational numbers in the interval

$$[-1 - 1/2^{j+1},\ 1 + 1/2^{j+1}],$$

multiples of $1/2^{j+1} 3^{n-1}$, and also the numbers dividing the intervals

$$[-1 - 1/2^j,\ -1 - 1/2^{j+1}]$$

and

$$[1 + 1/2^{j+1},\ 1 + 1/2^{j}]$$

into $3^n - 1$ equal parts (these conditions are given in the form needed for the inductive construction). In addition, it is required that

$$(\Phi_j)_t = e \quad \text{on } R^n/g_{j-1}I^n_{1+2^{-j}-1}.$$

We note that for $j = 0$ these conditions are satisfied for every δ-homeomorphism g.

Let

$$r_i = 1 + 1/2^{j-1} - (3^i - 1)/2^j(3^n - 1), \qquad 0 \le i \le n,$$

and

$$\bar{r} = (r_{i-1} + r_i)/2.$$

We shall call numbers of order (i, i'), in the case $i < i' \le n$: a) numbers, multiples of $1/2^j \cdot 3^{n-1}$, from $[-r_n, r_n]$, and b) numbers dividing the intervals $[-r_i, -r_n]$ and $[r_n, r_i]$ into $3^n - 1$ equal parts; and, in the case $1 \le i \le i'$: a) numbers, multiples of $1/2^{j+1} \cdot 3^{2n-i-1}$, from

$$[-r_n + 1/2^{j+1},\ r_n - 1/2^{j+1}],$$

b) numbers dividing into $3^{n-i}(3^n - 1)$ equal parts the intervals

$$[-r_n,\ -r_n + 1/2^{j+1}]$$

and

$$[r_n - 1/2^{j+1},\ r_n];$$

c) numbers dividing into 3^{n-i} equal parts each of the intervals

$$[-r_i, -r_n] \quad \text{and} \quad [r_n, r_i].$$

We construct Φ_j as the product $\psi_j\varphi_j$, where ψ_j is the product, taken in any order for all i, $1 \le i \le n$, and for all d of order (i, n), of the isotopies:

$$\xi_i(d - 2\varepsilon_{j-1},\ d - \varepsilon_{j-1},\ d - \varepsilon_j,\ d;\ r_n + \varepsilon_{j-1},\ r_0 - \varepsilon_{j-1})$$

$$\xi_i(d,\ d + \varepsilon_{j-1},\ d + \varepsilon_j,\ d + 2\varepsilon_{j-1};\ r_n + \varepsilon_{j-1},\ r_0 - \varepsilon_{j-1}),$$

and φ_j is the composition $\varphi_{j,n} \circ \cdots \circ \varphi_{j,1}$; moreover

$$(\varphi_{j,i})_1 g_{j-1} I_{i'}(d; r_i) \subset \operatorname{Int} I^{\varepsilon_{j-1}}_{i'}(d; r_i + \varepsilon_{j-1}),$$

where $1 \le i' \le n$ and d runs

moves points of order (i, i'). If one further requires that $\varphi_{j,i}$ depend continuously on g, they would be $(\eta_i/2^j)$-isotopies, where η_i does not depend on j or on g, and that

$$(\varphi_{j,i})_t = (\varphi_{j,i-1})_1 \quad \text{on} \quad R^n \setminus (\varphi_{j,i-1})_1 g_{j-1} I^n_{r_{i-1}},$$

then it is easy to see that $\psi_j\varphi_j$ satisfies all the requirements for Φ_j. In turn, φ_j is constructed as the product of the isotopies $\widetilde{\psi}_j$ and $\widetilde{\varphi}_j$, and also (for $i \to 1$) of the homeomorphism $(\varphi_{j,i-1})_1$. The isotopy $\widetilde{\psi}_j$ is the product, taken in any order for all $i' \neq i$ and for all numbers d of order (i, i'), of the isotopies

$$\xi(d''-2\varepsilon_{j-1},\, d'-\varepsilon_{j-1},\, d-\varepsilon_{j-1},\, d;\, \bar{r}+\varepsilon_{j-1},\, r_{i-1}-\varepsilon_{j-1})\, \xi_i'(d,\, d''+\varepsilon_{j-1},\, d+\varepsilon_{j-1},\, d''+2\varepsilon_{j-1};\, \bar{r}+\varepsilon_{j-1},\, r_{i-1}-\varepsilon_{j-1}),$$

where d' and d'' are the neighbors of d on the right and on the left among the points of order $(i-1, i')$. To construct $\widetilde{\varphi}_{j,i}$, take the points of order $(i-1, i')$ in $[-r_i, r_i]$, including the endpoints:

$$d_0, d_1, \dots, d_\lambda,$$

and let $d_{\lambda+1}$ also be the point of the same order neighboring r_i. The isotopy $\varphi_{j,i}$ is

$$\chi_{j,i,\chi} \circ \dots \circ \chi_{j,i,1},$$

where

$$(\chi_{j,i,k})_t = (\chi_{j,i,k-1})_1 \quad \text{outside} \quad g_{j-1}\Pi_i(d_{k-1}, d_{k+1}; r_{i-1}),$$

and

$$(\chi_{j,i,k})_1 = (\chi_{j,i,k-1})_1 \quad \text{outside} \quad g_{j-1}\Pi_i(d_{k-1}, d_k; r_{i-1}).$$

It follows that we can construct the $\chi_{j,i,k}$ independently of one another. Let $c_1, \dots, c_\lambda$ be the points of order (i, i) inside the interval $[d_{k-1}, d_k]$. We construct $\chi_{j,i,k}$ as the composition of the isotopies

$$\omega_{j,i,k,1} \circ \dots \circ \omega_{j,i,k,\lambda}$$

(first $\omega_{j,i,k,\lambda}$ is constructed), where

$$\omega_{j,i,k,l} = \sigma^{-1}\rho\sigma\tau((\omega_{j,i,k,l+1})_1,$$

and the isotopies σ, τ, ρ are described below. Let

$$\tilde{g} = (\omega_{j,i,k,l+1})_1(\chi_{j,i,k-1})_1(\varphi_{j,i-1})g_{j-1}$$

and put

$$H = \bigcup_{i' \neq i} \left(R^{\varepsilon_{j-1}}_{i, r_{i-1}} \cup R^{\varepsilon_{j-1}}_{i', -r_{i-1}} \right).$$

Let a_1 be the minimal number having the property that

$$\tilde{g} I_i(a_1; r_{i-1}) \subset R^+_{i, c_l - \varepsilon_{j-1}} \cup H,$$

and let a_2 be the maximal number such that, for all a', $0 < a' < a_2$,

$$\tilde{g} I^n_{r_{i-1}} \setminus \bigcup_{i' \neq i} \Pi_{i'}(-r_{i-1} + a',\, r_{i-1} - a';\, r_{i-1}) \subset \operatorname{Int} H.$$

Then $a_1(g)$ is a continuous function, and $a_2(g)$ is a lower semicontinuous function of g, moreover $a_2(g)$ is strictly positive. By the well-known theorem of Baire, it can be replaced by a continuous function $\tilde{a}_2(g)$. Let

$$\tau = \xi_i(d_{k-1},\, c_l,\, (a_1 + c_{l+1})/2,\, c_{l+1};\, r_{i-1} - \tilde{a}_2;\, r_{i-1})$$

and

$$\tau = \tilde{g}\,\tau\,\tilde{g}^{-1}.$$

Let b_1 and b_2 be the minimal numbers such that $b_1 \geq c$ and

$$(\tau)_1 \tilde{g} I_i(b_1; r_{i-1}) \subset R^+_{i, c_{l+1} - \varepsilon_{j-1}} \cup H,$$

$$(\tau)_1 \tilde{g} I_i(b_2; r_{i-1}) \subset R^+_{i, c_{l+2} - \varepsilon_{j-1}} \cup H.$$

Let

$$\sigma = \xi_i(b_1,\, c_{l+1},\, (b_2 + c_{l+2})/2,\, c_{l+2};\, r_{i-1},\, r_{i-1} + \varepsilon_{j-1})$$

and

$$\sigma = ((\tau)_1 \tilde{g})\sigma((\tau)_1 \tilde{g})^{-1}.$$

Finally, put

$$\rho = \xi_i(c_l - \varepsilon_{j-1},\, c_{l+1} - \varepsilon_{j-1},\, c_l + \varepsilon_{j-1},\, c_{l+2} - \varepsilon_{j-1};\, \bar{r} + \varepsilon_{j-1},\, r_{i-1} - \varepsilon_{j-1}).$$

4. Corollaries. The classes of stably equivalent homeomorphisms (see ([5])) coincide with the connected components and with the path-connected components of $\mathfrak{H}_m(M)$.

If a homeomorphism is approximated by stable ones, then it is itself stable.

The covering homotopy axiom is valid for embeddings with normal microbundles.

5. Questions. Does there exist, between two close piecewise linear homeomorphisms (say, of R^n), a small piecewise linear isotopy joining them?

Can the neighborhood O' be removed from the formulation (A)? Yes, if D is a locally flat submanifold, but I do not know the answer when D is a polyhedron or, say, a zero-dimensional compactum.

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Note: Figure translations are in progress. See original paper for figures.