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Abstract

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MATHEMATICS

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THE RADON-NIKODYM THEOREM AND THE REPRESENTATION OF VECTOR MEASURES BY AN INTEGRAL

(Presented by Academician P. S. Novikov on 7 VII 1967)

The paper considers a generalization of the Radon-Nikodym theorem to the case where both measures are vector-valued (with values in a Banach space). A similar situation was considered in paper ([1]), but in ([1]) rather severe requirements were imposed on vector measures. In the present paper some of these restrictions are removed; however, it is necessary to introduce a more general integral in order to represent one vector measure with respect to another (in ([1]) the bilinear Bartle integral is used for this purpose). As one of the applications of such a representation, a description will be given of vector measures of σ-finite variation with values in a reflexive space. In conclusion, necessary and sufficient conditions will be given for representing a vector measure by a Bochner integral.

By Σ we shall denote a certain σ-algebra of subsets of a set S. We shall call a countably additive function m, defined on Σ and taking values in a Banach space, a vector measure, i.e., m is such that for any sequence of pairwise disjoint sets

$$E_i \in \Sigma \quad \text{one has} \quad m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i),$$

where the series on the right-hand side converges unconditionally.

By the variation of the vector measure m on the set $E \in \Sigma$ (notation $v(m, E)$) we mean

$$\sup\left\{\sum_i \|m(E_i)\|\right\},$$

where the supremum is taken over all finite and countable systems $\{E_i\}$ of pairwise disjoint sets $E_i \subset E$, $E_i \in \Sigma$.

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If $v(m,S)<\infty$, then m is said to have finite variation. If

$$S=\bigcup_{n=1}^{\infty}E_n,$$

where $E_n\in\Sigma$ are such that $v(m,E_n)<\infty$, $n=1,2,...$, then we shall say that m has σ-finite variation.

It is said that a vector measure $m_2:\Sigma\to Y$ is absolutely continuous with respect to $m_1:\Sigma\to X$ ($m_2\ll m_1$), if for $E\in\Sigma$ from $v(m_1,E)=0$ it follows that $m_2(E)=0$.

Theorem 1. *In order that a vector measure m, absolutely continuous with respect to a positive measure μ, have σ-finite variation, it is necessary and sufficient that the set S can be represented in the form*

$$S=\bigcup_{N=1}^{\infty}E_N,$$

where $E_N\in\Sigma$ are such that if $F\subset E_N$, $F\in\Sigma$, then

$$\|m(F)\|\leq N\mu(F).$$

By π we shall denote any finite set of natural numbers; $\pi\geq\pi_1$ means that $\pi\supset\pi_1$. Let (Y,τ) be a separable locally convex space and let V be some neighborhood of zero in the topology τ.

Definition. A series $\sum_i y_i$ $(y_i\in Y,\ i=1,2,...)$ will be called *unconditionally summable with accuracy up to V to $y_0\in Y$* if there exists a π_0 such that for $\pi\leq\pi_0$

$$y_0-\sum_{i\in\pi}y_i\in V.$$

Consider a certain bilinear operator $u:Z\times X\to Y$, where X,Y,Z are arbitrary spaces. For brevity, instead of $u(z,x)$ we shall write zx. In what follows it is assumed that X is a Banach space, and that Y is endowed with a certain separable locally convex topology τ.

By a partition we shall mean any finite or countable family $\Delta=\{E_i\}$, $i=1,2,...$, of pairwise disjoint sets from Σ such that $\bigcup_i E_i=S$. If $\Delta'=\{F_j\}$, then $\Delta'\geq\Delta$ means that each set $F_j\in\Delta'$ is a subset of some $E_i\in\Delta$.

Definition. A function $f:S\to Z$ is called τ-integrable on S (with respect to $m:\Sigma\to X$), and its τ-integral on S is $y_S\in Y$, if for an arbitrary neighborhood

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of zero V in τ there is a partition Δ_V such that, for $\Delta \geq \Delta_V$, every series $\sum_i f(s_i)m(E_i)$ $(s_i \in E_i,\ E_i \in \Delta)$ is unconditionally summable with accuracy up to V to y_S.

Thus, by definition,

$$y_S = (\tau)\int_S f(s)\,dm = (\tau)\int_S f\,dm$$

(if the function f is τ-integrable on S, then the τ-integral is determined uniquely). It should be noted that the τ-integral depends, of course, on the bilinear operator $u : Z \times X \to Y$, and one ought to speak of the (τ, u)-integral with respect to m; but for simplicity we shall adhere to the terminology introduced above.

Definition. A function $f : S \to Z$ is called τ-integrable if, for every $E \in \Sigma$, the function $f(s)\chi_E(s)$ is τ-integrable on S (χ_E is the characteristic function of the set E). By definition,

$$(\tau)\int_E f\,dm = (\tau)\int_S f(s)\chi_E(s)\,dm.$$

If m is a positive measure, $Z = Y$, and $u(y, a) = ay$ $(a \in X,\ y \in Y)$, then our τ-integral coincides with Phillips' U-integral, see ([2]).

The following assertion is easily proved: if the functions f and g are τ-integrable, then the function $\alpha f + \beta g$ is τ-integrable (α, β are scalars), and for $E \in \Sigma$

$$(\tau)\int_E (\alpha f + \beta g)\,dm = \alpha \cdot (\tau)\int_E f\,dm + \beta \cdot (\tau)\int_E f\,dm.$$

Lemma. *Let $f : S \to Z$ be τ-integrable. If $\Delta = \{E_i\}$ is a partition of S such that, for $\Delta' = \{G_j\} \geq \Delta$, every series of the form $\sum_j f(s_j)m(G_j)$ $(s_j \in G_j)$ is unconditionally summable with accuracy up to V to $(\tau)\int f\,dm$, then for any $E \in \Sigma$ the series $\sum_i f(s_i)m(E_i \cap E)$ $(s_i \in E_i \cap E,\ E_i \in \Delta)$ is unconditionally summable with accuracy up to $3V$ to $(\tau)\int_E f\,dm$.*

Theorem 2 (on countable additivity of the τ-integral). *If the function $f : S \to Z$ is τ-integrable, then for any sequence of pairwise disjoint sets $E_i \in \Sigma$ and any neighborhood of zero V in τ there exists a π_V such that for every $\pi \geq \pi_V$ one has*

$$(\tau)\int_{\bigcup_{i=1}^{\infty} E_i} f\,dm - \sum_{i \in \pi} (\tau)\int_{E_i} f\,dm \in V.$$

Using the integral introduced, we shall give a generalization of the Radon-Nikodym theorem.

Let X^*, Y^* be the spaces dual respectively to the Banach spaces X, Y. Define the bilinear operator $u : L(X^*, Y^*) \times X^* \to Y^*$ as follows*:

$$u(z, x^*) = z(x^*), \quad z \in L(X^*, Y^*),\ x^* \in X^*.$$

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If, further, (Y^*, τ) is the space Y^* in its Y-topology (see ([3]), p. 453, definition 2), then the following holds

* By $L(X^*, Y^*)$ is denoted, as usual, the space of bounded linear operators from X^* into Y^*.

Theorem 3. Let $m_1 : \Sigma \to X^*$, $m_2 : \Sigma \to Y^*$ be vector measures, and let μ_1 be the variation of m_1. Suppose, further, that: 1) μ_1 is a full* finite measure on Σ; 2) the measure m_2 has σ-finite variation; 3) $m_2 \ll m_1$.

Then there exists on S a function f with values in $L(X^*, Y^*)$ such that

$$m_2(E) = (\tau) \int_E f \, dm_1, \quad E \in \Sigma.$$

Theorem 4. Let $m_1 : \Sigma \to X$, $m_2 : \Sigma \to Y$ (X, Y are Banach spaces) be vector measures for which the following conditions are satisfied: 1) $\nu(m_1, (\cdot))$ is a full finite measure on Σ; 2) the variation m_2 is σ-finite; 3) $m_2 \ll m_1$.

Then there exists on S a function f with values in $L(X, Y^{**})$ such that

$$m_2(E) = (\tau) \int_E f \, dm_1.$$

In this assertion the bilinear operator $u : L(X, Y^{**}) \times X \to Y^{**}$ is defined by the equality $u(z, x) = z(x)$, $x \in X$, $z \in L(X, Y^{**})$; for τ on Y^{**} the Y^*-topology is taken, and y is the image of the element $y \in Y$ under the natural embedding of Y into Y^{**}.

It should be noted that under the conditions of Theorem 4: a) the function f, generally speaking, is not uniquely determined, even up to a function equal to zero almost everywhere; b) it may happen that $f(s) \notin L(X, Y)$ for almost all $s \in S$. This, for example, will be the case when $Y = L_1[0,1]$, $S = [0,1]$, Σ is the collection of Lebesgue-measurable subsets of the interval $[0,1]$, $m_2(E) = \chi_E$ for $E \in \Sigma$, and m_1 is Lebesgue measure on Σ.

We give conditions sufficient for $f(s) \in L(X, Y)$ for all $s \in S$ in the case when m_1 is a scalar measure (for other conditions, also in the case when m_1 is a scalar measure, see (4), p. 269, Theorem 5).

Theorem 5. If, under the conditions of Theorem 4, m_1 is a positive measure and the required function f is almost separably valued, then there exists a separably valued Pettis-integrable** function $g : S \to Y$ such that

$$m_2(E) = P \int_E g(s) \, dm_1$$

for all $E \in \Sigma$.

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Let a positive full finite measure μ be given on Σ.

Theorem 6. If X is a reflexive space and the vector measure m ($m : \Sigma \to X$) is such that: 1) $m \ll \mu$; 2) the variation m is σ-finite, then there exists a separably valued Pettis-integrable function $f : S \to X$ such that

$$m(E) = (P) \int_E f \, d\mu.$$

Corollary 1. In order that a vector measure m, defined on Σ with values in a reflexive space X, absolutely continuous with respect to μ, have σ-finite variation, it is necessary and sufficient that there exist a function $f : S \to X$, Pettis-integrable, such that

$$m(E) = (P) \int_E f \, d\mu.$$

Corollary 2. If X is reflexive, then for every Pettis-integrable function $f : S \to X$ there exists a separably valued Pettis-integrable function $g : S \to X$ such that

$$(P) \int_E f \, dm = (P) \int_E g \, dm.$$

We give a theorem which is a generalization to the vector case of the result of G. P. Tolstov (see (6)). Let (S, Σ, μ) be a space with a full finite positive measure. We shall call a vector measure $\lambda : \Sigma \to X$ (X a Banach space) elementary (with respect to μ) if there exist $F \in \Sigma$ and $x \in X$ such that $\lambda(E) = x\mu(E \cap F)$ for every $E \in \Sigma$.

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* A nonnegative measure μ, defined on Σ, is called full if from $E \in \Sigma$, $F \subset E$, and $\mu(E) = 0$ it follows that $F \in \Sigma$.

** For the definition of Pettis and Bochner integrals see, for example, (5), Ch. III.

Theorem 6. In order that a vector measure $m : \Sigma \to X$ be representable as a Bochner integral of some function with respect to the measure μ, it is necessary and sufficient that there exist a sequence of elementary (with respect to μ) measures $\{\lambda_n\}$ ($\lambda_n : \Sigma \to X,\ n = 1, 2, ...$) such that

$$m(E) = \sum_{n=1}^{\infty} \lambda_n(E), \qquad \sum_{n=1}^{\infty} v(\lambda_n, S) < \infty.$$

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REFERENCES

[1] M. Rao, Proc. Nat. Acad. Sci. U. S. A., **15**, 771 (1964).
[2] R. Phillips, Trans. Am. Math. Soc., **47**, 114 (1940).
[3] H. Dunford, J. Schwartz, *Linear Operators*, Moscow, 1962.
[4] N. Dinculeanu, Vector Measures, Berlin, 1966.
[5] E. Hille, R. Phillips, *Functional Analysis and Semigroups*, Moscow, 1962.
[6] G. P. Tolstov, Matem. sborn., **71** (**113**), 420 (1966).

Note: Figure translations are in progress. See original paper for figures.

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