SMOOTH KNOTS IN A MANIFOLD OF THE HOMOTOPY TYPE OF A SPHERE

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Abstract

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SMOOTH KNOTS IN A MANIFOLD OF THE HOMOTOPY TYPE OF A SPHERE

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Let M^n be a smooth compact manifold. Denote by $\mathrm{Iso}(S^m, M^n)$ the set of smooth-isotopy classes of embeddings of the m-sphere S^m in M^n, and by $\mathrm{Iso}_p(S^m, M^n)$ the corresponding set in the combinatorial category. Haefliger ([1]) and Zeeman ([2]) computed these sets in the case $M^n = S^k \times D^{n-k}$, under the stability conditions for the group $\pi_m(S^k)$ and $k > 2,\ n-k > 2,\ m > 3,\ n-m > 2$. In the present note, under the same restrictions, we first study the set $\mathrm{Iso}^*(S^m, M^n)$ of combinatorial-isotopy classes of smooth embeddings of S^m in M^n, where M^n has the homotopy type of the sphere S^k. We then show that in the homotopically unstable region, for given m, n, k, this set depends on the differential type of M^n.

§ 1. Let $m > 3,\ n-m > 2$. Denote the n-dimensional disk by D^n and $\mathrm{Iso}(S^m, D^n)$ by C_m^{n-m}. It is known (see ([1])) that C_m^{n-m} is an abelian group.

Every embedding $i : D^n \subset M^n$ induces a map

$$i_* : C_m^{n-m} \longrightarrow \mathrm{Iso}(S^m, M^n).$$

Elements of $\mathrm{Im}\, i_* \subset \mathrm{Iso}(S^m, M^n)$ will be called local knots. Let M^n be simply connected. We shall define the action of $\mathrm{Im}\, i_*$ on $\mathrm{Iso}(S^m, M^n)$.

Let $\varphi \in \mathrm{Iso}(S^m, M^n)$, $\alpha \in \mathrm{Im}\, i_*$, and let $f_\varphi : S^m \subset M^n$, $f_\alpha : S^m \subset D_0^n \subset M^n$ be such representatives of φ and α that $\mathrm{Im}\, f_\varphi \cap D_0^n = \emptyset$. Join in M^n points $x \in \mathrm{Im}\, f_\varphi,\ y \in \mathrm{Im}\, f_\alpha$ by a smooth path not intersecting $\mathrm{Im}\, f_\varphi, \mathrm{Im}\, f_\alpha$, and take the connected sum $\mathrm{Im}\, f_\varphi \# \mathrm{Im}\, f_\alpha$ along a tube lying in an ε-neighborhood of this path. Since M^n is simply connected, $m > 3,\ n-m > 2$, it is clear that the operation described is possible for any φ, α, and the smooth-isotopy class of the resulting embedding $f_{\varphi+\alpha}$ depends only on φ and α. Let t be the coordinate on the interval I. We shall call a map $F_t : X \times I \to Y \times I$ fiberwise with respect to t if it commutes with the projection of the direct product onto the factor I.

Lemma 1. *Two elements of the set* $\mathrm{Iso}(S^m, M^n)$ *can be transformed one into the other by the group* $\mathrm{Im}\, i_*$ *if and only if their representatives are combinatorially isotopic.*

Proof. a) Let φ, α be as above, $f_\varphi, f_{\varphi+\alpha} : S^m \subset M^n$. Then there exists a disk $D^n \subset M^n$ outside which $\operatorname{Im} f_\varphi$ and $\operatorname{Im} f_{\varphi+\alpha}$ coincide, and $D^m \cap \operatorname{Im} f_\varphi$ and $D^n \cap \operatorname{Im} f_{\varphi+\alpha}$ are disks D_1^m, D_2^m in D^n, coinciding on the boundary:

$$\partial D_1^m = \partial D_2^m \subset \partial D^n.$$

By Theorem 1 of ([3]), D_1^m and D_2^m are combinatorially isotopic in D^n relative to the boundary. This isotopy extends to an isotopy

$$f_\varphi \to f_{\varphi+\alpha},$$

identical on the complement of the disk D_1^m.

b) Let $f, g : S^m \subset M^n$ be smooth embeddings; $D_1^m \subset S^m$, $D_2^m = S^m \setminus D_1^m$, and $f_t : f \to g$ a combinatorial isotopy. By Hirsch' s theorem on smoothing a contractible combinatorial submanifold ([4], Theorem B), f_t may be assumed smooth on $S^m \setminus \{*\}$, where $* \in D_2^m$. By Cerf' s theorem on the isotopy of D^n in M^n, by a smooth-isotopic replacement of g we may arrange that g and f coincide on D_1^m; in view of the simple connectedness of M^n, f_t may be regarded as fixed on D_1^m. Then the restriction of f_t to D_2^m determines a fiber-

with respect to t a combinatorial embedding $F_t : D_2^m \times I \subset M^n \times I$, smooth for $t = 0, 1$ and constant in t on $\partial D_2^m \times I$. According to Theorem B of [4], in $M^n \times I$ there exists a smooth closed neighborhood of the image $\operatorname{Im} F_t$, which is a disk $D_2^n \times I$, fiberwise embedded in $M^n \times I$ with respect to t, and moreover

$$\operatorname{Im} F_t \cap (\partial D_2^n \times I) = F_t(\partial D_2^m \times I).$$

Let $H_t : D_2^n \times I \to D^n \times I$ be a diffeomorphism, fiberwise in t, onto the standard disk such that $H_t \cdot F_t(\partial D_2^m \times I) \subset \partial D^n \times I$ is the product over I of the standard embedding S^{m-1} in S^{n-1}. Then the embeddings

$$H_0 \cdot F_0, \quad H_1 \cdot F_1 : \ (D_2^m, \partial D_2^m) \subset (D^n, \partial D^n)$$

represent elements $\alpha, \beta \in C_m^{n-m}$, and the image $\operatorname{Im} H_t \cdot F_t$ determines a combinatorial isotopy $\alpha \to \beta$. Add, as above, in $D^n \times (0)$ to $\operatorname{Im} H_0 \cdot F_0$ a local embedding $f_{\beta-\alpha} : S^m \subset M^n$. The new embedding

$$(H_0 \cdot F_0 \# f_{\beta-\alpha}) : D^m \subset D^n$$

is smoothly isotopic rel boundary to the embedding $H_1 \cdot F_1$. Therefore the embedding $\tilde{f} : S^m \subset M^n$, equal to f on D_1^m and to

$$H_0^{-1}(H_0 \cdot F_0 \# f_{\beta-\alpha})$$

on D_2^m, differs from f by a local knot and is smoothly isotopic to g in M^n. Thus,

$$\operatorname{Iso}(S^m, M^n) / \operatorname{Im} i_* = \operatorname{Iso}^*(S^m, M^n).$$

§ 2. **Lemma 2.** *Let M^{n+1} be a smooth compact manifold with boundary, having the homotopy type of the sphere S^{k+1}; let $\pi_1(\partial M^{n+1}) = 0$ and $k > 2,\ n-k > 2$. Then*

$$M^{n+1} = D_1^{n+1} \cup D_2^{n+1},$$

where

$$D_1^{n+1} \cap D_2^{n+1} = \partial D_1^{n+1} \cap \partial D_2^{n+1} = S^k \times D^{n-k},$$

and the embedding $j : S^k \times D^{n-k} \subset \partial D_1^{n+1}$ is canonical.

This lemma follows directly from Theorem 6.5 of [5].

We shall call a smooth embedding $f : S^{m+1} \subset M^{n+1}$ $(m > 2,\ n-m > 2)$ a suspension if

$$\operatorname{Im} f \cap (S^k \times D^{n-k}) = S^m.$$

It follows easily from Theorem 1 of [3] that suspensions over combinatorially isotopic embeddings of S^m in $S^k \times D^{n-k}$ are combinatorially isotopic in M^{n+1}.

It is known (see [1, 2]) that for $k, m, n-k, n-m > 2$, $\operatorname{Iso}(S^m, S^k \times D^{n-k})$ is an abelian group with respect to connected sum;

$$\operatorname{Iso}_p(S^m, S^k \times D^{n-k}) = \operatorname{Iso}^*(S^m, S^k \times D^{n-k})$$

and

$$\operatorname{Iso}(S^m, S^k \times D^{n-k}) = \operatorname{Iso}^*(S^m, S^k \times D^{n-k}) + C_m^{n-m}.$$

The mapping i_* in this case is a monomorphism, and the standard embedding

$$j : S^k \times D^{n-k} \subset S^n$$

induces a projection of $\operatorname{Iso}(S^m, S^k \times D^{n-k})$ onto C_m^{n-m} with kernel $\operatorname{Iso}^*(S^m, S^k \times D^{n-k})$. Therefore every element of $\operatorname{Iso}^*(S^m, S^k \times D^{n-k})$ is realized by a sphere $S^m \subset S^k \times D^{n-k}$ bounding a smoothly embedded disk D_1^{m+1} in D_1^{n+1}, but, possibly, not bounding such a disk in D_2^{n+1}. Thus we have, for every M^{n+1} satisfying Lemma 2 and $m > 2,\ n-m > 2$, a partial mapping

$$\Sigma : \operatorname{Iso}^*(S^m, S^k \times D^{n-k}) \to \operatorname{Iso}^*(S^{m+1}, M^{n+1}).$$

It is easy to see that the domain of definition of Σ is a subgroup $G_m(M^{n+1}) \subset \operatorname{Iso}^*(S^m, S^k \times D^{n-k})$. It coincides with the group $\operatorname{Iso}^*(S^m, S^k \times D^{n-k})$ if

$$2n > 3(m+1)$$

(and hence $C_m^{n-m} = 0$) or if $M^{n+1} = S^{k+1} \times D^{n-k}$; in the latter case Σ is a group homomorphism.

§ 3. **Theorem.** *Let M^{n+1} satisfy Lemma 2 and let $m > k > 3,\ n-m > 2$. Then the mapping*

$$\Sigma : g_m(M^{n+1}) \to \operatorname{Iso}^*(S^{m+1}, M^{n+1})$$

is an epimorphism for $m < 2k$ and an isomorphism for $m < 2k - 1$.

Proof. a) Let $f : S^{m+1} \subset M^{n+1}$ be a smooth embedding. It is easy to see that there exist smooth disks

$$(D_i^{n-k}, \partial D_i^{n-k}) \subset (M^{n+1}, \partial M^{n+1}), \quad i = 1, 2,$$

such that $D_i^{n-k} \subset D_i^{n+1}$,

$$D_i^{n-k} \cap S^k \times D^{n-k} = \emptyset$$

and $M^{n+1} \setminus D_i^{n-k}$ is contractible. Denote

$$\operatorname{Im} f \cap D_i^{n-k}$$

by M_i^{m-k}, and

$$M^{n+1} \setminus D_2^{n-k}$$

by E^{n+1},

$$\operatorname{Im} f \setminus D_2^{n-k} = \operatorname{Im} f \setminus M_2^{m-k}$$

by V^{m+1}. Let

$$U^{n+1} = D_1^{n-k} \times D_\varepsilon^k$$

be a closed tubular neighborhood of D_1^{n-k} in M^{n+1}. Denote

$$\operatorname{Im} f \cap U^{n+1}$$

by M_1^{m+1}; then M_1^{m+1} contracts to M_1^{m-k}. Since $m > 4,\ m < 2k$, the embedding M_1^{m-k} in V^{m+1} is homotopic to zero, and

$$\pi_l(\partial M^{m+1}) = \pi_l(V^{m+1})$$

for $l \le m - k$.

Inductively in l, one may assume M_1^{m+1} to be $(l-1)$-connected $(0 \le l \le m-k)$. Let $\lambda \in \pi_l(M_1^{m+1})$ be one of the generators. We realize λ by a sphere $S^l \subset \partial M_1^{m+1}$, and span in

$$V^{m+1} \setminus \operatorname{Int} U^{n+1}$$

over S^l an embedded disk D^{l+1}–this is possible in view of the condition $m < 2k$. The relative $(l+1)$-spheroid $(D^{l+1}$,

$S^l) \subset (E^{n+1}, U^{n+1})$ is null-homotopic, since $E^{n+1} \sim U^{n+1} \sim *$. Extend (D^{l+1}, S^l) to an embedding in $E^{n+1} \setminus \operatorname{Int} U^{n+1}$ of such a disk D^{l+2} that

$$\partial D^{l+2} = D^{l+1} \cup D_1^{l+1},$$

where $D_1^{l+1} = D^{l+2} \cap U^{n+1}$. Now attach to U^{n+1} a closed tubular neighborhood of the disk D^{l+2} in E^{n+1}. As a result we again obtain a disk; denote it by $\widetilde{U}^{n+1}$, and denote $\operatorname{Im} f \cap \widetilde{U}^{n+1}$ by $\widetilde{M}^{m+1}$. Now, if $\operatorname{Int} D^{l+2} \cap \operatorname{Im} f = \emptyset$, then $\widetilde{M}^{m+1}$

is obtained from M^{m+1} by attaching a handle with core disk D^{l+1}, which kills the generator $\lambda \in \pi_l(M^{m+1})$. The boundary $\partial\widetilde{M}^{m+1}$ of the manifold $\widetilde{M}^{m+1}$ is obtained from ∂M^{m+1} by a Morse modification killing the same generator λ in the group $\pi_l(\partial M^{m+1}) = \pi_l(M^{m+1})$. The condition $\operatorname{Int} D^{l+2} \cap \operatorname{Im} f = \emptyset$ is automatically satisfied for $l = 0$, since we have $n - m > 2$.

Let $l > 0$ and let

$$\Pi_1 = (D^{l+2} \setminus D^{l+1}) \cap \operatorname{Im} f$$

be a manifold with boundary

$$\partial\Pi_1 = \operatorname{Int} D_1^{l+1} \cap \operatorname{Im} f.$$

Then, in constructing $\widetilde{U}^{n+1}$, in addition to the handle P_λ^{l+1}, to M^{m+1} there is attached, along the embedding $\partial\Pi_1 \subset \partial M^{m+1}$, the pair $(\widetilde{\Pi}_1, \partial\widetilde{\Pi}_1)$—a closed tubular neighborhood $(\Pi_1, \partial\Pi_1)$ in $V^{m+1} \setminus \operatorname{Int} U^{n+1}$. The dimension of the manifold $\partial\Pi_1$, equal to $l+m-n+1$, is less than the connectivity of the manifold M^{m+1}; moreover, from the condition $m < 2k$ it follows that $2\dim\partial\Pi_1 < m$. Therefore $\partial\widetilde{\Pi}_1 \subset \partial M^{m+1}$ lies inside some disk $D_1^m \subset \partial M^{m+1}$. Consequently,

$$M^{m+1} \underset{\partial}{\cup} \Pi_1 \approx M^{m+1} \setminus (\Pi_1/\partial\Pi_1).$$

Now, inductively in i ($0 \le i \le \dim\Pi_1$), we shall kill the groups

$$\pi_i(\widetilde{M}^{m+1}) = \pi_i(\Pi_1/\partial\Pi_1)$$

as above. If in doing this no "second-order intersections"

$$\Pi_2 = (D^{l+2} \setminus D^{l+1}) \cap \operatorname{Im} f$$

arise, then as a result $\widetilde{M}^{m+1}$ will become $(l-1)$-connected. The group $\pi_l(\widetilde{M}^{m+1})$ is then obtained from the group $\pi_l(M^{m+1})$ by imposing the relation $\lambda = 0$. If, at some step of killing the homotopy of $\widetilde{\Pi}_1$, we have $\Pi_2 \neq \emptyset$, then we begin to kill the homotopy of its tubular neighborhood $\widetilde{\Pi}_2$ in the same way as was done with $\widetilde{\Pi}_1$. In doing so a "third-order intersection" Π_3 may arise.

In the general case there arises a chain of successive intersections:

$$\Pi_1,\ \Pi_2,\ \Pi_3, \ldots$$

It is finite, since from the condition $n - m > 2$ it follows that

$$\dim \Pi_{s+1} < \dim \Pi_s.$$

Therefore the killing of the group $\pi_l(M^{m+1})$ is carried out in a finite number of steps. After the groups $\pi_j(M^{m+1})$, $j \leq m-k$, have been killed, the manifold $\widetilde{M}^{m+1}$ becomes contractible, and $\partial \widetilde{M}^{m+1}$ becomes $(m-k)$-connected. From the condition $m > k > 3$ and Smale' s theorem on a pair ((5), Theorem 5.1) it follows that $\widetilde{M}^{m+1} = D^{m+1}$.

Now $\widetilde{U}^{n+1} = D^{n+1}$ intersects $\operatorname{Im} f$ in the disk D^{m+1} and coincides with the original U^{n+1} on ∂M^{n+1}. By Cerf' s theorem on the isotopy of D^n in M^n, there exists a diffeotopy Φ_t of the manifold M^{n+1} onto itself, carrying $\widetilde{U}^{n+1}$ to D_1^{n+1}. The restriction Φ_t to $\operatorname{Im} f$ carries f into a concordant embedding.

b) Let $f_{0,1} = \Sigma g_{0,1}$ be representatives of the element $\varphi \in \operatorname{Iso}^*(S^{m+1}, M^{n+1})$. According to Lemma 1, the embeddings $f_{0,1}$ can be chosen smoothly isotopic. Let

$$f_t : S^{m+1} \times I \subset M^{n+1} \times I$$

be a levelwise in t smooth embedding determined by the isotopy $f_0 \to f_1$. By a smooth isotopy of $\operatorname{Im} f_t$ in $M^{n+1} \times I$ relative to the "boundary" $(t = 0, 1)$, we arrange, as above, that the condition

$$(S^k \times D^{n-k} \times I) \cap \operatorname{Im} f_t = S^m \times I$$

be satisfied. In doing so, the embedding

$$\tilde{g}_t : S^m \times I \subset S^k \times D^{n-k} \times I$$

will not be levelwise in t for $0 < t < 1$. But, by Smale's theorem on the extension of a diffeomorphism ((5), Theorem 3.2) and Cerf' s theorem on pseudoisotopy ((6)), $\tilde{g}_t$ is isotopic in $S^k \times D^{n-k} \times I$ relative to the "boundary" $(t = 0, 1)$ to an embedding g_t that is levelwise in t. This gives us an isotopy

$$g_0 \to g_1.$$

From this theorem follows, in particular, Zeeman' s result:

$$\operatorname{Iso}^*(S^m, S^k \times D^{n-k}) = \operatorname{Iso}^*(S^{m+1}, S^{k+1} \times D^{n-k})$$

for $m > k > 2$, $n - m > 2$, $m \leq 2k - 1$ (see (2)).

§ 4. We shall show by an example that, in the case when the group $\pi_m(S^k)$ is unstable, the set $\mathrm{Iso}^*(S^m, M^n)$, where $M^n \sim S^k$, depends on the differential type of M^n.

Let χ be the Hopf fibration over S^4 with fiber D^4; $\eta = \chi \oplus 3$. Denote by M^8, M^{11} the total spaces of the fibrations χ and η; then M^8 and M^{11} satisfy Lemma 2. The smooth embedding $f : S^7 = \partial M^8 \subset M^8 \subset M^{11}$ represents a generator β of the free part of the group $\pi_7(S^4) = Z + Z_{12}$. We shall prove that β is not realized even by a combinatorial embedding of S^7 in $S^4 \times D^7$. Indeed, the suspension $\Sigma : \pi_7(S^4) \to \pi_8(S^5)$ is an epimorphism, under which β goes to the generator $\Sigma\beta \subset \pi_8(S^5) = Z_{24}$, and if β were realized by a combinatorial embedding of S^7 in $S^4 \times D^7$, then $\Sigma\beta$ would be realized by a combinatorial embedding of S^8 in $S^5 \times D^7$, i.e. the map

$$\pi : \mathrm{Iso}_P(S^8, S^5 \times D^7) \to \pi_8(S^5)$$

would be an epimorphism. But, according to (2), $\mathrm{Iso}_P(S^8, S^5 \times D^7) = \pi_6(S^3) = Z_{12}$, and such an epimorphism is impossible. Moreover, the embedding $f : S^7 \subset M^{11}$ is not homotopic and, consequently, not isotopic to a suspended one.

Thus, there is no natural, i.e. projection-commuting into the group $\pi_7(S^4)$, isomorphism between the sets $\mathrm{Iso}^*(S^7, S^4 \times D^7)$ and $\mathrm{Iso}^*(S^7, M^{11})$, and the suspension $\Sigma : g_6(M^{11}) \to \mathrm{Iso}^*(S^7, M^{11})$ is not an epimorphism.

Analogous examples can be obtained from the Hopf fibration over S^8 with fiber D^8.

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Note: Figure translations are in progress. See original paper for figures.