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MATHEMATICS

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Abstract

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MATHEMATICS

V. V. VELICHENKO

ON SUFFICIENT CONDITIONS FOR OPTIMALITY IN THE MAXIMUM PRINCIPLE

(Presented by Academician L. S. Pontryagin, February 15, 1968)

Using the method of dynamic programming, V. G. Boltyanskii showed ([1,2]) that the maximum principle of L. S. Pontryagin in the theory of optimal processes, established at first as a necessary condition for optimality ([3,4]), is a sufficient condition if a regular synthesis of control is carried out.

In the present note it is shown that the sufficient conditions for optimality formulated in ([1,2]) can be obtained directly by means of the formalism of the maximum principle. It is shown that the optimality of a synthesis carried out in accordance with the necessary conditions of the maximum principle is ensured by the continuity of the field of extremals with respect to their initial data. The case of the fulfillment of conditions of a sufficiently general form at the right end of the trajectory is considered.

1°. Formulation of the problem. Let an n-dimensional system of equations be given

$$\dot{x}(t) = f(x, u, t) \tag{1}$$

and a functional

$$I(x, u) = \Phi[x(T), T], \tag{2}$$

defined as a function of the right end $\{x(T), T\}$ of the trajectory of system (1). It is required to find a control $u(t) \in U$ minimizing the functional (2) under the condition that the point $\{x(T), T\}$ satisfies the conditions

$$M_j[x(T), T] = 0, \quad j = 1, 2, \dots, m \leq n. \tag{3}$$

It is assumed that the functions $\Phi(x, t)$ and $M_j(x, t)$ are continuous together with their first-order partial derivatives and have bounded second-order partial derivatives with respect to all variables; the vector function $f(x, u, t)$ has the

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indicated properties with respect to the variables x, u and is continuous in t; the domain U is closed and bounded, and the control $u(t)$ is piecewise continuous.

2°. Necessary conditions of L. S. Pontryagin' s maximum principle in the form of Lagrange multipliers. Construct the auxiliary functional

$$\Psi[x(T), T] = \lambda_0 \Phi[x(T), T] + \sum_{j=1}^{m} \lambda_j M_j[x(T), T], \tag{4}$$

where $\lambda_i,\ i = 0, 1, \dots, m$, are certain numbers. Define the function $H(x, p, u, t)$ and the vector function $p(t)$ by the conditions

$$H(x, p, u, t) \equiv (p, f(x, u, t)), \tag{5}$$

$$\dot{p}(t) = -\operatorname{grad}_x H(x, p, u, t), \quad p(T) = -\operatorname{grad}_x \Psi[x(T), T].$$

It can be shown (see, for example, ([5])) that the necessary conditions for optimality in the posed problem with fixed left end $\{x(T_0), T_0\}$ of the trajectory of system (1) are as follows: there exist such–

which are not all simultaneously zero, $\lambda_i,\ i = 0, 1, \dots, m,\ \lambda_c \geq 0$, such that

$$H(x, p, u, t) = \sup_{v \in U} H(x, p, v, t), \qquad T_0 < t < T, \tag{6}$$

$$-H[x(T), p(T), u(T-0), T] + \partial\Psi[x(T), T]/\partial T = 0. \tag{7}$$

3°. Sufficient conditions for optimality. Suppose that through each point $\{x, t\}$ of some domain $A \subset X \times t$ there passes an extremal–a trajectory of system (1) satisfying the necessary optimality conditions (3), (5)–(7). Consider those pairs of points $\{x_0', T_0'\}$, $\{x_0'', T_0''\}$ of the domain A that can be connected by admissible (for $u(t) \in U$) trajectories of system (1) wholly belonging to A.

We shall say that the totality of extremals forms an L-continuous field in A if, for every $\varepsilon > 0$, there exist such bounded constants α and β that, for any two extremals $x'(t)$ and $x''(t)$, passing respectively through the points $\{x_0', T_0'\}$ and $\{x_0'', T_0''\}$, such that $|T_0' - T_0''| \leq \varepsilon$, the corresponding controls $u'(t)$ and $u''(t)$ and terminal times T' and T'' satisfy the conditions

$$\int_{\max\{T_0', T_0''\}}^{\min\{T', T''\}} |u'(t) - u''(t)|\, dt \leq \alpha\varepsilon, \qquad |T' - T''| \leq \beta\varepsilon.$$

Theorem 1. *If the field of extremals in A is L-continuous and on each extremal $\lambda_0 > 0$, then each extremal wholly belonging to A, for any of its points taken as*

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fixed initial data at the left endpoint, delivers in A an absolute minimum to the functional (2), subject to the conditions (3).

Proof is carried out by comparing the values of the functional Ψ (which coincide, by virtue of the fulfillment of conditions (3), with the values of the functional Φ) for the extremals of the field and arbitrary admissible trajectories. The comparison is made with the aid of the formula for the variation of the functional (4) ([6],[7]).

4°. Optimality of regular synthesis. Theorem 1 can be used to obtain optimality conditions for a synthesizing control $u(x,t)$.

For practical applications, it is convenient to use sufficient optimality conditions, in the form of the maximum principle, for regular synthesis, established by V. G. Boltyanskii with the aid of the method of dynamic programming ([1],[2]). A direct verification shows that a field of extremals satisfying the conditions of regular synthesis is L-continuous. Therefore Theorem 1 makes it possible to prove the sufficiency of the conditions of regular synthesis directly with the aid of the formalism of the maximum principle.

Denote by M the $(n+1-m)$-dimensional terminal manifold determined by the conditions (3). In the case considered here, $\{x(T),T\} \in M$. We shall call a synthesis for problem (1)–(3) **regular** if:

1) the regularity conditions for the synthesis formulated in ([1],[2]) are fulfilled (with the obvious inessential changes connected with the use of the manifold M as the terminal set and with the formulation of the necessary conditions of the maximum principle in the form (4)–(7));

2) the extremals do not touch the manifold M, and on each of them $\lambda_0 > 0$ (in the case $m = n$, this condition follows from condition 1)).

Theorem 2. *If in some domain A there is a regular synthesis for problem (1)–(3), then each extremal wholly belonging to A, for any of its points taken as fixed initial data at the left endpoint, delivers in A an absolute minimum to the functional (2), subject to the conditions (3).*

5°. On the nonuniqueness of the synthesizing control.
When constructing a control synthesis in accordance with the necessary conditions of the maximum principle, the function $u(x,t)$ sometimes turns out to be nonunique. This is a consequence of the fact that extremals of different families pass through each point of a certain domain $B \subset A$. In the examples known to the author, such a situation arises in the following cases:

a) Extremals are reflected from a certain manifold F (the reflection surface), which is part of the boundary of the domain B. Through each point $\{y,\tau\} \in B$ there pass a reflected extremal (having a common point with F for $t \in [\tau,T]$) and an unreflected extremal (having no common points with F for $t \in [\tau,T]$).

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b) In the domain B there pass extremals of families terminating on different parts of the terminal manifold (3).

In case a), the use of Theorems 1 and 2 permits one to assert that *if the reflected extremal passing through the point* $\{y, \tau\}$ *lies entirely in the domain where the unreflected extremals satisfy the conditions of Theorem 1 or Theorem 2, then the value of the functional (2) for it cannot be smaller than for the unreflected extremal passing through* $\{y, \tau\}$. This property of reflected extremals makes it possible, as a rule, not to regard them as candidates for optimality and thereby to eliminate the nonuniqueness of $u(x, t)$ associated with reflection of extremals.

In case b), the construction of a single-valued function $u(x, t)$ reduces to the construction of a surface (or surfaces) on which Φ takes identical values for extremals of different families. Such a construction gives a surface that is an element of the set N ([1,2]).

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Moscow Institute of Physics and Technology

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Note: Figure translations are in progress. See original paper for figures.

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