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Abstract

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A PRIORI ESTIMATES OF SOLUTIONS OF THE DIRICHLET PROBLEM FOR SOME HYPOELLIPTIC EQUATIONS

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Let

$$P(x, D_x)u \equiv \sum_{|\alpha| \le m} a_\alpha(x) D_x^\alpha u \equiv$$

$$\equiv \sum_{\alpha_1+\ldots+\alpha_n \le m} a_{\alpha_1\ldots\alpha_n}(x_1, \ldots, x_n) \frac{\partial^{\alpha_1+\cdots+\alpha_n}}{(ix_1)^{\alpha_1} \ldots (ix_n)^{\alpha_n}} u \tag{1}$$

be a hypoelliptic operator of constant strength (^{1}) of order m, defined on functions of n real variables $u(x) = u(x_1, \ldots, x_n)$, given in a bounded domain Ω with continuously differentiable boundary Γ. The coefficients of the operator $a_\alpha(x)$ are continuous functions in $\Omega + \Gamma$.

By a solution of the Dirichlet problem we mean a sufficiently smooth solution $u(x)$ of the equation

$$P(x, D_x)u = f, \qquad x \in \Omega, \tag{2}$$

for which, at each boundary point $y \in \Gamma$, all derivatives up to an order less than a certain number m_+, depending on the operator P and the point y, vanish. If A_y is the matrix of such a rotation of the coordinate system $A_{yx} = \tilde{x}$ under which the directions of the axis $\tilde{x}_n$ and of the inward normal at the point $y \in \Gamma$ coincide, and A'_y is the transposed matrix, then $m_+(y)$ is equal to the number of roots with positive imaginary part of the polynomial $P(y, A'_y\xi)$, solved with respect to ξ_n, for sufficiently large values of $|\xi'| = (\xi_1^2 + \cdots + \xi_{n-1}^2)^{1/2}$ with real $\xi_1, \ldots, \xi_{n-1}$ $(0 \le m_+(y) \le m)$.

Suppose that there exist such mutually orthogonal directions $l_1, \ldots, l_s$ that, after a rotation $B'x = \tilde{x}$ aligning the directions of the axes $\tilde{x}_1, \ldots, \tilde{x}_s$ with $l_1, \ldots, l_s$, the polynomial $P(y, B'\xi)$ has order m in each of the variables $\xi_1, \dots, \xi_s$, and for some constants C_1, C_2 the conditions

$$\xi_j^m / |P(y, B'\xi)| < C_1 \quad \text{for } |\xi| > C_2 \text{ and } j = 1, \dots, s. \tag{3}$$

are satisfied.

Let $n(y)$ be the unit inward normal at the point $y \in \Gamma$, and let $N(y)$ be the projection of $n(y)$ onto the subspace spanned by the vectors l_j,

$$N(y) = \sum_{j=1}^{s} (n(y), l_j) l_j.$$

We require that:

$$\begin{gathered}\text{the function } |N(y)| \text{ defined on } \Gamma \text{ have zeros} \\ \text{of order not higher than the first.}\end{gathered} \tag{4}$$

Except for the set $\{y \in \Gamma,\ N(y) = 0\}$, Γ is assumed to be m times continuously differentiable, with uniform estimates of the derivatives.

Theorem. *Under the imposed restrictions (1), (3), (4), the solution of the Dirichlet problem for equation (2) admits the estimate*

$$\|Q(x, D_x)u\|_{\mathcal{L}_2(\Omega)} \le K(\|f\|_{\mathcal{L}_2(\Omega)} + \|u\|_{\mathcal{L}_2(\Omega)}), \tag{5}$$

where the constant K does not depend on u, and $Q(x, D_x)$ is any operator weaker than $P(x, D_x)$, i.e., with some constants C_1 and C_2,

$$|Q(x, \xi)/P(x, \xi)| < C_1 \quad \text{for } |\xi| > C_2. \tag{6}$$

The method of proof is a generalization of the usual method of local estimates, and the main attention is directed to the choice of a special cut-off function that makes it possible to pass to local considerations.

Let us note, first, that the function u can be represented as the sum of two functions, one of which is concentrated in an ε_1-neighborhood of Γ with sufficiently small ε_1, while the support of the other lies inside Ω. For the function concentrated in Ω, the estimate we need is already known ([1]) (Theorem 7.4.2), and therefore it is enough to consider functions concentrated in an ε_1-neighborhood of Γ.

In a neighborhood of a point $y \in \Gamma$ introduce local coordinates $\tilde{x}$ with origin at the point y and with the axis $\tilde{x}_n$ directed along the interior normal, $\tilde{x} = A_y(x - y)$. Let $u(x) = u(\tilde{x}, y)$. Then equation (2) can be written in the form

$$P(y, A'_y D_{\tilde{x}}) u(\tilde{x}, y) = f_1 \equiv f + [P(y, D_x) - P(x, D_x)]u. \tag{2'}$$

Let $\zeta(\tilde{x}, y)$, for fixed $y \in \Gamma$, be a function of $\tilde{x}$, smooth in Ω. We shall impose several restrictions on the form of this function, and for the moment we require that the following property hold.

Property 1. As a function of $\tilde{x}$, $\zeta(\tilde{x}, y)$ is concentrated in a $2\varepsilon_1$-neighborhood of the point $y \in \Gamma$, and the intersection $\nu(y)$ of its support with the boundary Γ lies in an $\varepsilon(y) = \varepsilon_2 |N(y)|$-neighborhood of the point y. The constant ε_2 must be so small that in the $\varepsilon(y)$-neighborhood of the point y the transformation defined for each y in the $\varepsilon(y)$-neighborhood of y as a displacement in the direction $N(y)$, under which $\gamma(y)$ passes into the plane tangent to Γ at the point y, is uniformly bounded in the C^m norm with respect to $y \in \Gamma$. Such a choice $0 < \varepsilon_2 \leq \varepsilon_1$ is possible owing to the smoothness of the boundary Γ.

For each y, the displacement defined in the $\varepsilon(y)$-neighborhood can be extended by a displacement with preservation of the uniform boundedness of the C^m-norm to the whole space and, in particular, to the support of the function $\zeta(\tilde{x}, y)$.

Applying this change of variables $\tilde{x} = R(z)$, we transform the operator $P(y, A'_y D_{\tilde{x}})$ to the form

$$P(y, A'_y D_z) + \sum_{j=1}^{r} c_j(y, z) P_j(y, D_z)$$

with some finite r, continuous coefficients $c_j(y, z)$, $c_j(y, 0) = 0$, and operators $P_j(y, D_z)$, each of which has coefficients constant with respect to z and is weaker than $P(y, A'_y D_z)$ (this is ensured by condition (3)).

Considering the function $v(z, y) = u(R(z), y)\zeta(R(z), y)$, we can then write equation $(2')$ as

$$P(y, A'_y D_z) v = f_2, \qquad z_n > 0, \tag{7'}$$

where

$$f_2 \equiv f_1 \zeta + \sum_{0<|\alpha|\leq m} \frac{1}{\alpha!} \left[D_\xi^\alpha P(y, A'_y \xi) \right]_{\xi = D_x} u(\tilde{x}, y)\, D_x^\alpha \zeta(\tilde{x}, y) - \sum_{j=1}^{r} c_j(y, z) P_j(y, D_z) v. \tag{7''}$$

The function v satisfies the boundary conditions

$$D_{z_n}^j v\big|_{z_n=0} = 0, \qquad j = 0, \dots, m_+ - 1.$$

The function v, solving problem $(7'), (7''')$, can be written through

$$\tilde{f}_{2+}(\xi) = \int_{z_n>0} f_2(z) e^{-i\xi z}\, dz,$$

$$v(z, y) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} d\xi\, e^{i\xi z_n} \tilde{f}_{2+}(\xi)/P(y, A'_y \xi).$$

Let us now note that the variables $(y, z' = 0, z_n) = (y, z_n)$ define a coordinate system in a sufficiently small ε_1-neighborhood of the boundary Γ, and the transformation $(y, z_n) \leftrightarrow x$ is one-to-one, smooth, with Jacobian separated from zero and infinity.

Property 2 of the function $\zeta(\tilde{x}, y)$: in the variables z, in an ε_1-neighborhood of the boundary, for $z' = 0$ all derivatives of the function ζ vanish, while the function itself is equal to 1.

If $Q(x, D_x)$ is an operator with bounded coefficients which is weaker than $P(x, D_x)$, then, after replacing the variables x by z, y according to the formula $z = R(A_y(x - y))$, Q is transformed into $Q_1(z, y, D_z)$ with bounded coefficients and weaker than $P(y, A'_{yD} z)$. Therefore, with certain bounded functions $b_j(y, z)$ $(j = 1, \dots, r)$,

$$Q_1(z, y, D_z) = \sum_{j=1}^{r} b_j(z, y) P_j(y, D_z)\ (^1).$$

By property 2 of the function ζ,

$$\sum_{j=1}^{r} b_j(z, y) P_j(y, D_z) v(z, y)\Big|_{z'=0} = Q(x, D_x) u\Big|_{x \to (y, z_n)}.$$

Therefore, instead of estimating $\|Q(x, D_x)u\|_{\mathcal{L}_2(\Omega)}$, it is enough for us to be able to estimate $\|P_j(y, D_z)v|_{z'=0}\|_{\mathcal{L}_2(y, z_n)}$ $(j = 1, \dots, r)$.

It is not difficult to obtain the estimate

$$\|P_j(y, D_z)v|_{z'=0}\|_{\mathcal{L}_2(y, z_n)} \leq K_1 \|f\|_{\mathcal{L}_2(y, z_n)}. \tag{8}$$

In order to pass from inequality (8) to the required

$$\|P_j(y, D_z)v|_{z'=0}\|_{\mathcal{L}_2(y, z_n)} \leq K_2 \|f\|_{\mathcal{L}_2(y, z_n)} \leq K \|f\|_{\mathcal{L}_2(\Omega)},$$

we apply the usual arguments using the small variation of continuous coefficients in a small domain and the possibility of obtaining the estimate (8) for any operators weaker than $P(x, D_x)$.

Here the main role is played by the estimate

$$\left\| \left\{ [D_\xi^\alpha P(y, A'_y\xi)]_{\xi=D_x}\, u(\tilde{x}, y) \right\} D_x^\alpha \zeta(\tilde{x}, y) \Big|_{\substack{\tilde{x}\to z\\ z'=0}} \right\|_{\mathcal{L}_2(y,z_n)} \le K_3 \|f\|_{\mathcal{L}_2(y,z_n)},$$

which we derived from the inequality

$$\sup_{y,\xi_n} \left\| \frac{D_\xi^\alpha P(y, A'_y\xi)}{P(y, A'_y\xi)} \int_{-\infty}^{\infty} e^{-iz\xi} D_z^\alpha \zeta \prod_{j=1}^{n} \frac{\sin L_1 z_j}{z_j}\, dz \right\|_{\mathcal{L}_2(|\xi'|>L_2-|\xi_n|)} \le L_3 < \infty \qquad (9)$$

with certain constants L_1, L_2, L_3.

Inequality (9) may be regarded as **property 3** of the function ζ. The requirement (4) of the theorem is a rather rough sufficient condition for (9) to hold.

Boundary-value problems for certain special types of hypoelliptic equations are the subject of the works ($^{2-5}$).

Let us note here that for parabolic equations (^{3,4}) and for the special type of boundary considered in these works, it is also possible to construct a function ζ with properties 1, 2, 3 and, consequently, to obtain the estimate (5).

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