ON GÅRDING' S INEQUALITY FOR DEGENERATE ELLIPTIC OPERATORS

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Abstract

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Let Ω be a bounded domain of the $(n+1)$-dimensional Euclidean space of points $x = (x_1, \dots, x_n, x_{n+1})$. Let m be a fixed natural number. Suppose that for every $z \in \Omega$ there is given a real homogeneous form $a(z, \xi)$ of the real variables $\xi_1, \dots, \xi_n, \xi_{n+1}$ of degree $2m$, with: 1) the coefficients of the form sufficiently smooth and uniformly continuous; 2) $\inf_{|\xi|=1} a(z, \xi)$ having a positive lower bound in Ω. With every such form one can associate the Dirichlet form

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(z) D^\alpha u D^\beta u, \tag{1}$$

where the coefficients $a_{\alpha\beta}(z)$ are real, symmetric, satisfy condition 1), and, moreover,

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(z) \delta^\alpha \delta^\beta = a(z, \xi). \tag{2}$$

As is known, such forms exist. If $m = 1$, then there exists only one Dirichlet form, and this form is positive definite. For $m > 1$ it may happen that no Dirichlet form is positive definite. From some Dirichlet form belonging to the form $a(z, \xi)$ we construct the Dirichlet integral

$$I(u, u) = \int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(z) D^\alpha u D^\beta u \, dz. \tag{3}$$

The following assertion is valid.

Gårding' s inequality (1). There exist positive constants C_1 and C_2 such that

$$I(u, u) \geq C_1 \|u\|^2_{W_2^m} - C_2 \|u\|^2_{L_2}. \tag{4}$$

This inequality, as is known, plays an important role in the use of functional methods in the theory of partial differential equations.

In the present note we indicate a class of degenerate elliptic operators for which Gårding' s inequality (^{1}) retains its validity. Let E^+_{n+1} denote the half-space $y > 0$ ($x_{n+1} = y$) of the Euclidean $(n+1)$-dimensional space of points $z = (x, y)$ ($x = (x_1, \dots, x_n)$). We shall consider a bounded domain Ω^+ lying in the half-space $y > 0$ and adjacent to the hyperplane $y = 0$. Let $C_0^\infty(\Omega^+)$ denote the set of infinitely differentiable functions having compact support contained in Ω^+. On this set we consider the differential operator

$$\widetilde{D}^k_y u = y^k \frac{\partial^k u}{(y\partial y)^k} = \frac{\partial^k u}{\partial y^k} + \sum_{i=1}^{k} C_i^{(k)} \frac{\partial^{k-i} u}{y^i \partial y^{k-i}}. \tag{5}$$

For every point $z \in \Omega^+$ we prescribe a real homogeneous form $a(z; \xi, \eta)$ of the real variables $\xi_1, \dots, \xi_n, \eta$ of degree $2m$. In this case

we shall assume that the coefficients satisfy conditions 1) and 2) and, as $y \to 0$, behave in a definite manner (see below).

To each such form we associate a Dirichlet form of the type

$$\sum_{|\alpha|+k=m} \sum_{|\beta|+l=m} a^{kl}_{\alpha\beta}(z)\, \widetilde{D}^{\alpha+k} u\, \widetilde{D}^{\beta+l} u, \tag{6}$$

where $\widetilde{D}^{\alpha+k} = D^\alpha_x \widetilde{D}^k_y$, $\widetilde{D}^{\beta+l} = D^\beta_x \widetilde{D}^l_y$; the coefficients $a^{kl}_{\alpha\beta}$ are real, symmetric, satisfy the conditions indicated above, and possess the property that

$$\sum_{|\alpha|+k=m} \sum_{|\beta|+l=m} a^{kl}_{\alpha\beta}(z)\, \xi^\alpha \tau^k \xi^\beta \tau^l = a(z; \xi, \tau). \tag{7}$$

Let us take one of these Dirichlet forms and, from it, form an integral of Dirichlet type

$$\tilde{I}_\gamma(u, u) = \int_{\Omega^+} \sum a^{kl}_{\alpha\beta}(z)\, \widetilde{D}^{\alpha+k} u(z)\, \widetilde{D}^{\beta+l} u(z)\, y^{2\gamma}\, dz. \tag{8}$$

If the coefficients are constant, then in the Fourier–Bessel images the form (8) is written as follows:

$$\tilde{I}_\gamma = C^{(m)}_\gamma \int_{E^+_{n+1}} a(s, t) \left| \tilde{f}_{\gamma-\frac{1}{2}}(s, t) \right|^2 t^{2\gamma}\, ds\, dt, \tag{9}$$

where $a(s, t)$ is a homogeneous form of degree $2m$ with constant coefficients, and $\tilde{f}_{\gamma-\frac{1}{2}}$ is the mixed Fourier–Bessel transform (see (4)). To obtain a lower estimate for the forms just indicated, we introduce the corresponding functional space, considered by the author earlier in papers (^{2}–^{4}). The space $\widetilde{W}^{k}_{y,2,\gamma}(\Omega^{+})$ is defined as the closure of $C_0^\infty(\Omega^+)$ in the norm

$$\|u\|^2_{\widetilde{W}^{k}_{y,2,\gamma}} = \int_{\Omega^+} |u|^2 y^{2\gamma}\,dz + \int_{\Omega^+} |\widetilde{D}^k_y u|^2 y^{2\gamma}\,dz. \tag{10}$$

We shall characterize differential properties in all the remaining directions as follows. Let l_i be an integer nonnegative number. The functional spaces $W^{l_i}_{x_i,2,\gamma}(\Omega^+)$ $(i = 1, 2, \dots, n)$ are defined as the closure of $C_0^\infty(\Omega^+)$ in the usual S. L. Sobolev norm with the weight factor $y^{2\gamma}$. The space $\widetilde{W}^{l,k}_{x,y,2,\gamma}(\Omega^+)$ is defined as the intersection of the corresponding spaces with the norm determined by the formula

$$\sum_{i=1}^{n} \|u\|^2_{W^{l_i}_{x_i,2,\gamma}(\Omega^+)} + \|u\|^2_{\widetilde{W}^{k}_{y,2,\gamma}(\Omega^+)}. \tag{11}$$

This space has many remarkable properties, and for it, in terms of Fourier–Bessel images, there is an equivalent norm (see (4)).

If on $C_0^\infty(\Omega^+)$ we introduce the scalar product

$$(u,v)^\gamma_j = \int_{\Omega^+} \sum_{|\beta|+k=j} \widetilde{D}^{\beta+k}u(z)\,\widetilde{D}^{\beta+k}v(z)\,y^{2\gamma}\,dz \quad (j = 0, 1, 2, \dots, m) \tag{12}$$

and complete the set $C_0^\infty(\Omega^+)$ in the norm

$$\|u\|^2_j = (u,u)^\gamma_j + (u,u)^\gamma_0, \tag{13}$$

then we obtain a Hilbert space $\overset{\circ}{H}{}^{\gamma}_{j}$ with norm equivalent to the norm of $\widetilde{W}^{m,m}_{x,y,2,\gamma}$. Using the apparatus of the mixed Fourier–Bessel transform, we prove the following assertion:

Theorem 1. *There exist positive constants* c_1 *and* c_2 *such that*

$$\tilde{I}_\gamma(u,u) \geq C_1\|u\|^2_{W^m_{2,\gamma}} - C_2\|u\|^2_{L_{2,\gamma}}. \tag{14}$$

Let us now consider the ordinary Dirichlet form of the type

$$\sum_{|\alpha|+k=m} \sum_{|\beta|+l=m} a^{kl}_{\alpha\beta}(z) D^{\alpha+k}u(z) D^{\beta+l}u(z) \tag{15}$$

with all the preceding restrictions on the coefficients $a^{kl}_{\alpha\beta}$, and, in addition, assume that they satisfy the inequality

$$\left|a^{kl}_{\alpha\beta}(z)\right| \le Cy^{k+l-2} \qquad (k, l \ge 1). \tag{16}$$

For this form we construct the Dirichlet integral

$$\tilde{I}_\gamma(u,u) = \int_{\Omega^+} \sum a^{kl}_{\alpha\beta}(z) D^{k+\alpha}u(z) D^{\beta+l}u(z) y^{2\gamma}\, dz. \tag{17}$$

Then the following assertion holds.

Theorem 2. *There exist positive constants C_1' and C_2' such that*

$$\tilde{I}_\gamma(u,u) \ge C_1'\|u\|^2_{W^m_{2,\gamma}} - C_2'\|u\|^2_{L_{2,\gamma}}. \tag{18}$$

Upper estimates in inequalities (14) and (18) are obtained quite simply. Let us now consider the following problem. Let $\mathcal{L}$ be a formally self-adjoint operator of order $2m$,

$$\mathcal{L} = (-1)^m \sum_{|\alpha|+k=|\beta|+l\le m} D^{\alpha+k}\left(y^{2\gamma} a^{kl}_{\alpha\beta}(z) D^{\beta+l}u(z)\right) \tag{19}$$

with coefficients satisfying the restrictions indicated above. The non-self-adjoint case can be reduced to this one.

Dirichlet problem. In the domain Ω^+ a function $g \in H^\gamma_m(\Omega^+)$ is given, and an infinitely differentiable function h such that $\|h\|_{L_{2,\gamma}}$. Find a solution u of the equation

$$\mathcal{L}u = h$$

such that $g - u \in \dot{H}^\gamma_m$. The Dirichlet problem is solvable for arbitrary g and h, if the homogeneous problem has only the zero solution. The pair of equations $Qu = h$ and $Q^*v = g$, where Q^* is the operator formally adjoint to Q, forms a Fredholm pair. One can study the existence and properties of solutions of the equation

$$Qu - \lambda u = 0,$$

belonging to H^γ_m, etc.

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Note: Figure translations are in progress. See original paper for figures.

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