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ON TRACE FORMULAS FOR SELF-ADJOINT OPERATORS

MATHEMATICS

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Abstract

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MATHEMATICS

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ON TRACE FORMULAS FOR SELF-ADJOINT OPERATORS

(Presented by Academician L. S. Pontryagin, 13 XII 1967)

Let A be a self-adjoint operator with discrete spectrum; let $\lambda_1, \dots, \lambda_n, \dots$ be the eigenvalues of A, each eigenvalue being counted as many times as its multiplicity; denote by $e_1, \dots, e_n, \dots$ the corresponding eigenvectors. The operator A is such that

$$\sum_k \frac{1}{\lambda_k} < \infty \qquad (\lambda_k \neq 0). \tag{1}$$

Consider the operator $C = A + B$, where B is a bounded self-adjoint operator. As is known, C also has a discrete spectrum $\mu_1, \dots, \mu_n, \dots$. In the work of Halberg and Kramer ([2]) the following theorem is proved.

If

$$\sum_{k=1}^{\infty} (\mu_k - \lambda_k) < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} (Be_k, e_k) < \infty$$

then

$$\sum_{k=1}^{\infty} (\mu_k - \lambda_k) = \sum_{k=1}^{\infty} (Be_k, e_k). \tag{2}$$

In our paper a somewhat stronger condition than (1) is imposed on A, but it is proved that, under this condition, for the existence of the right-hand side of equality (2) it is necessary and sufficient that its left-hand side exist. In addition, the question of the relation between

$$\sum_{k=1}^{\infty} (Be_k, e_k) \quad \text{and} \quad \sum_{k=1}^{\infty} (Bf_k, f_k)$$

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for orthonormal bases $\{e_k\}_1^\infty$ and $\{f_k\}_1^\infty$ that are close in a certain sense is considered.

For simplicity of exposition we shall assume the operator A to be positive definite, with the λ_n arranged in increasing order. By condition (1), there exists a sequence of neighboring pairs $\lambda_{n_i}, \lambda_{n_i+1}$ such that

$$\frac{1}{\lambda_{n_i+1} - \lambda_{n_i}} \longrightarrow 0.$$

Introduce the notation $\omega_n = \lambda_{n+1} - \lambda_n$; $\nu_n = (\lambda_n + \lambda_{n+1})/2$.

Theorem 1. *For any $n \geq 2$ there exists an M such that, for sufficiently large N_i, the inequality*

$$\left| \sum_{k=1}^{N_i} (\mu_k - \lambda_k) - \sum_{k=1}^{N_i} (Be_k, e_k) \right| \leq M \left(1 + \frac{\nu_{N_i}}{\omega_{N_i}} \right) \sum_{k=1}^{\infty} \frac{1}{|\nu_{N_i} - \lambda_k|}.$$

In what follows we shall omit the index i.

Theorem 1 follows from several preliminary lemmas, which are proved by computing and estimating contour integrals.

Lemma 1.

$$\int_{C_R} \lambda \operatorname{Sp} R_\lambda^A (BR_\lambda^A)^n (E + BR_\lambda^A)^{-1} \, d\lambda \to 0$$

as $R \to \infty$, where $R_\lambda^A = (A - \lambda E)^{-1}$, and C_R is an arc of the circle $|\lambda| = R$; $\operatorname{Re} \lambda \leq v_N$.

Lemma 2.

$$\left| \int_{v_N - i\infty}^{v_N + i\infty} \lambda \operatorname{Sp} R_\lambda^A (BR_\lambda^A)^{n+1} (E + BR_\lambda^A)^{-1} \, d\lambda \right| \leq A_1 \frac{v_N}{\omega_N^n} \sum_{k=1}^{\infty} \frac{1}{|v_N - \lambda_k|}.$$

Lemma 3.

$$\left| \int_{C_N} \operatorname{Sp} (BR_\lambda^A)^n \, d\lambda \right| \leq A_2 \sum_{k=1}^{N} \frac{1}{v_N - \lambda_k},$$

where C_N is a contour containing the points $\lambda_1, \ldots, \lambda_N$; $n \geq 2$.

Lemma 4.

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$$\int_{C_N} \operatorname{Sp}(BR_\lambda^A)\,d\lambda = -2\pi i \sum_{k=1}^{N} (Be_k, e_k).$$

The lemma is obvious.

Lemma 5.

$$\int_{C_N} \lambda \operatorname{Sp} R_\lambda^A (BR_\lambda^A)^n\,d\lambda = -\frac{1}{n} \int_{C_N} \operatorname{Sp}(BR_\lambda^A)^n\,d\lambda.$$

Proof of the theorem. Choose the contour $C_{R,N}$ in the complex plane as follows: the arc of the circle $|\lambda| = R$, $\operatorname{Re}\lambda \le v_N$, and the segment cut out by this circle on the line $\operatorname{Re}\lambda = v_N$. For sufficiently large N this contour contains the points $\lambda_1, \dots, \lambda_N$ and $\mu_1, \dots, \mu_N$, since $|\mu_n - \lambda_n| \le \|B\|$. We shall next use the identity

$$\sum_{k=1}^{N} (\mu_k - \lambda_k) = \frac{1}{2\pi i} \int_{C_{R,N}} \lambda \operatorname{Sp}(R_\lambda^A - R_\lambda^C)\,d\lambda, \tag{3}$$

where $R_\lambda^C = (C - \lambda E)^{-1}$. But

$$R_\lambda^C = R_\lambda^A (E + BR_\lambda^A)^{-1},$$

$$(E + BR_\lambda^A)^{-1} = \sum_{k=1}^{n} (-1)^k (BR_\lambda^A)^k + (-1)^{n+1} (BR_\lambda^A)^{n+1} (E + BR_\lambda^A)^{-1}. \tag{4}$$

Substituting (4) into (3) and using Lemmas 1-5, we obtain our theorem.

Corollary. If

$$\left(1 + \frac{v_{N_i}}{\omega_{N_i}^n}\right) \sum_{k=1}^{\infty} \frac{1}{|v_{N_i} - \lambda_k|} \to 0 \quad \text{as } N_i \to \infty,$$

then for the existence of the sum

$$\sum_{k=1}^{\infty} (\mu_k - \lambda_k)$$

it is necessary and sufficient that the sum

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$$\sum_{k=1}^{\infty}(Be_k, e_k)$$

exist, and in that case they are equal.

Example. If $\lambda_n = O(n^\alpha)$ and the multiplicity of λ_n is $O(n^\beta)$, then it can be shown that in this case our theorem gives

$$\left|\sum_{k=1}^{N}(\mu_k - \lambda_k) - \sum_{k=1}^{N}(Be_k, e_k)\right| \le M\frac{\ln N}{N^{n[(\alpha-\beta)-1]-1}}.$$

if, however, $\alpha - \beta > 1$, then, by virtue of the arbitrariness of n, the assertion of the corollary holds.

Thus, in order to compute

$$\sum_{k=1}^{\infty}(\mu_k - \lambda_k),$$

in many cases it is sufficient to be able to compute

$$\sum_{k=1}^{\infty}(Be_k, e_k),$$

where $\{e_k\}_1^\infty$ is a proper basis of the operator A. Therefore the question of the connection between the matrix traces of the operator B in various orthonormal bases of Hilbert space is of interest.

Let $\{e_k\}_1^\infty$ and $\{f_k\}_1^\infty$ be two orthonormal bases such that

$$\sum_{k=1}^{\infty}\|f_k - e_k\|^2 < \infty.$$

Let $Ue_k = f_k,\ U = E + S$, where S is a Hilbert-Schmidt operator.

Theorem 2.

$$\sum_{k=1}^{n}(Be_k, e_k) - \sum_{k=1}^{n}(Bf_k, f_k) = \sum_{k=1}^{n}([SB]e_k, e_k) + o(1).$$

Proof.

$$\sum_{k=1}^{n}(Be_k, e_k) - \sum_{k=1}^{n}(Bf_k, f_k) = \sum_{k=1}^{n}(Be_k, e_k) - \sum(U^{-1}BUe_k, e_k)$$
$$= -\sum_{k=1}^{n}((S^*B + BS + S^*BS)e_k, e_k),$$

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since $U^{-1} = E + S^*$. The operator S^*BS is nuclear; therefore

$$\mathrm{Sp}(S^*BS) = \mathrm{Sp}(SS^*B);$$

$$UU^{-1} = E = E + S + S^* + SS^*;$$

whence, in view of the fact that

$$\sum_{k=1}^{n}(S^*BSe_k, e_k) = \sum_{k=1}^{n}(SS^*Be_k, e_k) + o(1) = -\sum_{k=1}^{n}((SB + S^*B)e_k, e_k) + o(1),$$

our theorem follows.

Corollary 1. If the matrix trace $[SB]$ exists in the basis $\{e_k\}_1^\infty$, then for the convergence of

$$\sum_{1}^{\infty}(Bf_k, f_k)$$

it is necessary and sufficient that the series

$$\sum_{k=1}^{\infty}(Be_k, e_k)$$

converge, and in this case

$$\sum_{k=1}^{\infty}(Bf_k, f_k) = \sum_{k=1}^{\infty}(Be_k, e_k) + \sum_{k=1}^{\infty}([BS]e_k, e_k).$$

Corollary 2. If $f_n = e_n + g_n + \varepsilon_n$, where

$$\sum_{k=1}^{\infty}\|g_k\|^2 < \infty \quad \text{and} \quad \sum_{k=1}^{\infty}\|\varepsilon_k\| < \infty,$$

i.e., if $S = T_1 + T_2$, where T_2 is a nuclear operator, then in the assertion of Corollary 1 one may replace S by T_1.

Indeed,

$$[BS] = [BT_1] + [BT_2],$$

but

$$\mathrm{Sp}[BT_2] = \sum_{k=1}^{\infty}([BT_2]e_k, e_k) = 0.$$

Example. Consider the Sturm-Liouville problem

$$-y'' + \lambda y = 0, \qquad y'(0) - hy(0) = 0, \qquad y'(\pi) + Hy(\pi) = 0.$$

The eigenvalues of this problem have the asymptotics

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$$\lambda_n = n^2 + \frac{H+h}{n\pi} + O\left(\frac{1}{n^2}\right).$$

The eigenfunctions have the asymptotics:

$$f_n(x) = \sqrt{\frac{2}{\pi}}\left\{\cos nx + \frac{h}{n}\sin nx - \frac{H+h}{n\pi}x\sin nx\right\} + O\left(\frac{1}{n^2}\right).$$

Let μ_n be the eigenvalues of the problem

$$-y'' + q(x)y + \mu y = 0$$

under the same boundary conditions.

Here the operator B is multiplication by $q(x)$, where $q(x)$ is continuously differentiable and its mean value is equal to 0. In order to compute

$$\sum_{k=1}^{\infty}(Bf_n, f_n),$$

we use Corollary 2.

Put

$$e_n(x) = \sqrt{\frac{2}{\pi}}\cos nx, \qquad T_1 e_n(x) = \sqrt{\frac{2}{\pi}}\frac{\sin nx}{n}\left[h - \frac{H+h}{\pi}x\right],$$

$$\sum_{k=1}^{\infty}(Bf_n, f_n) = \sum_{k=1}^{\infty}(Be_k, e_k) + \sum_{k=1}^{\infty}([BT_1]e_k, e_k),$$

but it can be shown that

$$\sum_{k=1}^{\infty}([BT_1]e_k, e_k) = 0,$$

whence we obtain

$$\sum_{k=1}^{\infty}(\mu_k - \lambda_k) = \frac{2}{\pi}\sum_{k=1}^{\infty}\int_0^{\pi} q(x)\cos^2 nx\,dx = \frac{2}{\pi}\sum_{k=1}^{\infty}\int_0^{\pi} q(x)\cos 2nx\,dx =$$

$$= \frac{q(0)+q(\pi)}{4},$$

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which is well known ([1]).

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named after M. V. Lomonosov

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CITED LITERATURE

[1] I. M. Gel' fand, B. M. Levitan, DAN, 88, No. 4 (1953). [2] C. J. A. Halberg, V. A. Kramer, Duke Math. J., 27, No. 4 (1960).

Note: Figure translations are in progress. See original paper for figures.

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