ON FEATHERED PARACOMPACTS

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Abstract

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ON FEATHERED PARACOMPACTS

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In (^{1,2}) A. V. Arhangel' skii introduced the concept of a feathered space (= p-space). To the class of feathered paracompacts one has succeeded in extending a number of theorems concerning metric spaces, paracompact spaces complete in the sense of Čech, and bicompacts.* Here we shall present some results of this type.

Theorem 1. *Let $f : X \to Y$ be a closed mapping**; let X be a feathered paracompact. Then

$$Y = \bigcup_{i=0}^{\infty} Y_i,$$

where Y_i is discrete for $i = 1, 2, ...$, and $f^{-1}(y)$ is a bicompact if $y \in Y_0$.*

For some special cases the assertion of the theorem can be obtained trivially from results contained in ($^{3-6}$). The problem in this formulation was first considered by K. Morita in (^{7}) for the quite simple case of paracompact locally bicompact spaces. A. V. Arhangel' skii (^{8,9}) and N. S. Lašnev (^{10}) proved the assertion respectively for weakly paracompact spaces complete in the sense of Čech and for metric spaces.

Let us turn to the proof of Theorem 1.

I. In the space X there exists a countable family of locally finite coverings*** γ_n, $n = 1, 2, ...$, satisfying the following conditions:

1^0. $[\gamma_n x] \subseteq \Gamma_{n-1} \in \gamma_{n-1}$ for every $x \in X$****.

2^0. $\bigcap_n \gamma_n x$ is a bicompact for every $x \in X$.

3^0. The sets $\gamma_n x$ form a neighborhood base of the bicompact $\bigcap_n \gamma_n x$.

4^0. If $x_n \in \gamma_n x$, then the sequence $\{x_n\}$ has limit points***** in $\bigcap_n \gamma_n x$ and only there.

II. Let $f(x_n) = f(x'_n) = y_n$, where the points $y_n \in Y$ are pairwise distinct. Then the set $\{x_n\}$ is discrete in X if and only if the set $\{x'_n\}$ is discrete.

Indeed, by virtue of the closedness and continuity of the mapping f, either of these sets is discrete if and only if the set $\{y_n\}$ is discrete.

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* All these spaces are p-paracompacts.

** Only continuous mappings of completely regular spaces are considered. "Covering" always means "open covering."

*** The existence of such coverings follows, for example, from the following considerations. Every p-paracompact can be perfectly mapped onto a metric space Y (see (^{1,2})). The inverse images of a sequence of locally finite open coverings of the space Y, twice star-refined into one another, evidently have the required properties. We note that condition 4^0 follows from conditions 1^0–3^0.

**** By γM, where γ is some family of sets and M is a set, is meant the star of the set M with respect to the family γ, i.e. the union of all elements of γ intersecting M.

***** We recall that a point x is called a limit point for a sequence $\{x_n\}$ if every neighborhood of the point x contains points x_n for an infinite set of indices n. In particular, if the point x occurs in the sequence an infinite number of times, then it is a limit point of the sequence.

III. For each $x \in X$, denote by M_x* the set of those points $x' \in X$ for which there exists a countable set $\{y_n\} \subseteq Y$ of pairwise distinct points of the space Y such that $\gamma_n x \cap f^{-1}(y_n) \neq \emptyset$ and $\gamma_n x' \cup f^{-1}(y_n) \neq \emptyset$ for every n.

The definition of the set M_x is symmetric in the sense that $x' \in M_x \Leftrightarrow x \in M_{x'}$. It is not difficult to show that M_x is bicompact, but we shall not need this below.

Let the points $y_n \in Y$ be pairwise distinct, $x_n \in f^{-1}(y_n)$, $x'_n \in f^{-1}(y_n)$, $\gamma_n x \ni x_n$, and let x' be a limit point of the set $\{x'_n\}$. We shall show that $x' \in M_x$. From the sequence $\{x'_n\}$ pass, if necessary, to a subsequence $\{x'_{m(n)}\}$, where $m(n) > n$, $m(n_1) \neq m(n_2)$ when $n_1 \neq n_2$, and $\gamma_n x' \ni x'_{m(n)}$. Then

$$x_{m(n)} \in \gamma_n x \cap f^{-1}(y_{m(n)}), \qquad x'_{m(n)} \in \gamma_n x \cap f^{-1}(y_{m(n)}),$$

which proves the inclusion $x' \in M_x$.

IV. Let $\{x_n\} \subseteq X$, $x'_n \in M_x$. The sequence $\{x_n\}$ is discrete if and only if the sequence $\{x'_n\}$ is discrete.

Let the point x be a limit point of the sequence $\{x_n\}$. We may assume that $x_n \in \gamma_{n+1} x$. Otherwise pass to a subsequence.

Choose pairwise distinct points $y_n \in Y$,

$$\gamma_{n+1}x_n \cap f^{-1}(y_n) \neq \emptyset, \qquad \gamma_{n+1}x'_n \cap f^{-1}(y_n) \neq \emptyset.$$

Let $x''_n \in \gamma_{n+1}x_n \cap f^{-1}(y_n) \subseteq \gamma_{n+1}x_n \subseteq \gamma_n x$, $x'''_n \in \gamma_{n+1}x'_n \cap f^{-1}(y_n)$. The set $\{x''_n\}$ has a limit point in $\bigcap_n \gamma_n x$, therefore (by II) there exists a point x', a limit point of the set $\{x'''_n\}$. Then $\gamma_{n+1}x' \ni x'''_{m(n)}$ for some $m(n) > n$, and $\gamma_n x' \supseteq \gamma_{n+1}x'''_{m(n)} \supseteq \gamma_{m(n)+1}x'''_{m(n)} \ni x'_{m(n)}$.

By I, in $\bigcap_n \gamma_n x'$ there is a point that is a limit point of the sequence $\{x'_n\}$, which, thus, is not discrete. The second part of assertion IV follows from what has just been proved, in view of the inclusion $x_n \in M_{x'_n}$.

V. We shall now show that the set Y_m of points of the space Y whose inverse images are not contained in any element of the family $\sigma_m = \{\gamma_m M_x,\ x \in X\}$** is discrete. If this is not so, then in X there is a point x that is a limit point for the set $f^{-1}(Y_m \setminus \{f(x)\})$. This means, in particular, that for every n there is a $y_n \in Y_m$ such that $\gamma_n x \cap f^{-1}(y_n) \neq \emptyset$, $f^{-1}(y_n) \setminus \gamma_m M_x \neq \emptyset$, where the points y_n are pairwise distinct. Choose points $x_n \in f^{-1}(y_n) \cap \gamma_n x$ and $x'_n \in f^{-1}(y_n) \setminus \gamma_m M_x$. The set $\{x_n\}$ has a limit point in $\bigcap_n \gamma_n x$; therefore, by I, the set $\{x'_n\}$ also has a limit point, which, by III, lies in M_x and at the same time, by virtue of the closedness of $X \setminus \gamma_m M_x$, lies in $X \setminus \gamma_m M_x$. The contradiction obtained proves the discreteness of the set Y_m.

VI. We shall show that $f^{-1}(y)$ is bicompact if $y \in Y_0 = Y \setminus \bigcup_{m=1}^{\infty} Y_m$. For each n there exists a point x'_n such that $f^{-1}(y) \subseteq \gamma_n M_{x_n}$. Take arbitrarily $x' \in f^{-1}(y)$ and $x'_n \in \gamma_n x' \cap M_{x_n} \neq \emptyset$. The sequence $\{x'_n\}$ has limit points in $\bigcap_n \gamma_n x'$, and therefore the sequence $\{x_n\}$ is not discrete.

Let $\{x''_n\} \subseteq f^{-1}(y)$, $x'''_n \in \gamma_n x''_n \cap M_{x_n}$. The sequence $\{x'''_n\}$ is not discrete and there exists a point x'', a limit point of $\{x'''_n\}$. Passing, if need, to a subsequence $\{x''_{m(n)}\}$ such that $m(n) > n$, $\gamma_{n+1}x'' \supset x_{m(n)}$. Then $\gamma_n x'' \supset \gamma_{n+1}x''_{m(n)} \supset \gamma_{m(n)}x''_{m(n)} \supset x_{m(n)}$. The set $\{x''_n\}$ has a limit point in $\bigcap_n \gamma_n x''$, which, by virtue of the closedness of $f^{-1}(y)$, lies in $f^{-1}(y)$. We have proved the compactness of $f^{-1}(y)$. But a compact paracompact space is bicompact—the theorem is proved.

Corollary 1. *Under a closed mapping of a paracompact p-space, the cardinality of the set of points of the image whose inverse images are not bicompact does not exceed either the weight of the space being mapped or the weight of the image.*

Corollary 2. *A closed image of a paracompact p-space is everywhere, except for some σ-discrete set of its points, a space of point-countable type (see* ([12,13])*).*

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* The set M_x may also be empty.

** The family σ_m is not required to cover X.

In Theorem 1 one cannot dispense with the requirement that the space X be feathered. Here is a simple example. Define the space θ as follows. Take a certain set of points of cardinality continuum and one further, specially distinguished point τ. We shall call a set closed if it is finite or contains the point τ.* Consider the product $\theta \times I$, where I is the ordinary interval. We obtain the paracompact $X \subset \theta \times I$ by throwing out the end-points of each interval $a \times I$, where $a \in \theta$ is an isolated point. Then the natural projection $\pi : X \to \theta$ is a closed mapping, but nevertheless the set of points whose inverse images are not bicompact—the set of all isolated points of the space θ—is certainly not σ-discrete.**

We shall now extend to p-paracompacts a statement proved in (11) by A. S. Mishchenko for the case of bicompacts, making essential use here of his constructions (see the lemma).

Theorem 2. *Let X be a paracompact p-space and B its base. If the cardinality of the set of elements of the base B containing an arbitrary point $x \in X$ does not exceed $\aleph_\lambda$, then in X there exists a base which decomposes into some set of locally finite families, the cardinality of which does not exceed $\aleph_\lambda$.*

The proof of the following assertion is textually the same as the proof of the theorem from (11).

Lemma. *Let B be some family of subsets of the set X, whose cardinality at each point does not exceed $\aleph_\lambda$. Then the cardinality of the set of finite minimal*** *covers of the set X by elements of the family B does not exceed $\aleph_\lambda$.*

Since X is a p-paracompact, we can map X perfectly onto some metric space Y. In Y there is a base $\bar{B}$ decomposing into a countable set of locally finite families γ_n $(n = 1, 2, \ldots)$. By the lemma, the cardinality of the set N_Γ of finite minimal covers of the form $\{G_1, \ldots, G_s\}$ (where $G_1, \ldots, G_s \in B$) of the inverse image Γ of an arbitrary element of the base $\bar{B}$ does not exceed $\aleph_\lambda$. We enumerate the set N_Γ by transfinite numbers smaller than $\omega\aleph_\lambda$. Denote by $\delta_{n\alpha}$ the family of sets of the form $\mathcal{G} \cap \Gamma$, where $\Gamma \in f^{-1}\gamma_n$, and $G \in B$ is an element of that cover of the set Γ belonging to N_Γ which received the number α. It is easy to see that the family $\delta_{n\alpha}$ is locally finite. $\delta = \bigcup_{n,\alpha} \delta_{n\alpha}$ is some base of the space X. Indeed, let $x \in G_x \subset X$, $G_x \in B$. For every point $x' \in f^{-1}f(x)$ we find $G_{x'} \in B$, $x' \in G_{x'}$, $x \in G_{x'}$ for $x \neq x'$. From this cover of the bicompact $f^{-1}f(x)$ we choose a finite minimal one $\{G_1, \ldots, G_s\}$; G_x, obviously, is an element of this cover. The inverse images of elements of $\bar{B}$ form a base of inverse images of points from Y, therefore there is a Γ—the inverse image of

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* The space θ is, obviously, the simplest hereditarily feathered, hereditarily paracompact nonmetrizable bicompact.

** The first example of this kind was constructed by J. Wattel.

*** A cover is called minimal if it contains no proper subcover.

an element B—such that

$$f^{-1}f(x) \subseteq \Gamma \subseteq \bigcup_{i=1}^{s} G_i.$$

Then $G_x \cap \Gamma \in \delta_{n\alpha}$ (for some n and α), $x \in G_x \cap \Gamma \subseteq G_x$.

An immediate consequence of what has been proved is

Theorem 3*. *A paracompact p-space with a point-countable base is metrizable.*

I am pleased to express my gratitude to my adviser A. V. Arhangel' skii.

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* This is the complete solution of A. V. Arhangel' skii' s problem ([13]). Somewhat earlier V. I. Ponomarev obtained a solution of this problem for the case of finally compact paracompact p-spaces. It relied on the technique of perfect irreducible mappings.

Note: Figure translations are in progress. See original paper for figures.

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