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# A FIXED-POINT THEOREM

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**Abstract**

**Full Text**

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**A FIXED-POINT THEOREM**

**FOR GENERALIZED CONDENSING OPERATORS**

*(Presented by Academician A. N. Tikhonov, 25 III 1968)*

In <sup>(1)</sup> a fixed-point theorem was proved for one general class of operators acting in Banach spaces. It was shown that this class includes contraction operators, completely continuous operators, and also sums of operators of the two indicated types. It is natural to wish to obtain a more general fixed-point principle which, in particular, would also contain the classical principle of A. N. Tikhonov <sup>(2)</sup>. The result obtained along these lines is presented in the present paper.

**Definition.** Let  $E$  be a locally convex linear topological space. A continuous operator  $f$  acting in  $E$  will be called **generalized condensing** on a set  $T \subseteq E$  if, for any set  $\Omega \subseteq T$ , from  $f(\Omega) \subseteq \Omega$  and from the bicomcompactness of the set  $\Omega \setminus \overline{\text{co}}f(\Omega)$  there follows the bicomcompactness of  $\overline{\Omega}$ .

**Theorem.** Let  $T(\subseteq E)$  be a convex closed set. Suppose that a generalized condensing operator  $f$  maps  $T$  into itself:  $f(T) \subseteq T$ . Then  $f$  has at least one fixed point in  $T$ .

**Proof.** We recall some facts concerning generalized sequences. A generalized sequence of elements of a set  $X$  is a mapping  $g : D \rightarrow X$  of a directed set  $D$  into  $X$  (a directed set is a partially ordered set in which every finite subset has a majorant). A point  $x$  of a topological space  $X$  is called a generalized limit point of the generalized sequence  $g : D \rightarrow X$  if, for every neighborhood  $U$  of the point  $x$  and every  $d_0 \in D$ , there exists  $d \geq d_0$  such that  $g(d) \in U$ . Every generalized sequence in a bicomcompact space  $X$  has at least one generalized limit point (see <sup>(3)</sup>, pp. 38-41).

Let  $\alpha$  be an ordinal number;  $A_\alpha$  the set of all ordinal numbers not exceeding  $\alpha$ . We shall prove by induction that there exists a generalized sequence  $g : A_\alpha \rightarrow T$  possessing the following properties:

- a)  $g(\beta) = f[g(\beta - 1)]$ , if  $\beta$  is an ordinal number of the first kind;
- b)  $g(\beta)$  is a generalized limit point of the generalized sequence  $g : A^\beta \rightarrow T$  ( $A^\beta$  is the set of all ordinal numbers preceding  $\beta$ ), if  $\beta$  is an ordinal number of the second kind.

If  $\alpha = 0$ , then the existence of such a sequence is obvious. Suppose it exists for every  $\alpha < \gamma$ . We shall show that it exists also for  $\alpha = \gamma$ .

If  $\gamma$  is an ordinal number of the first kind, then we may extend the generalized sequence  $g : A_{\gamma-1} \rightarrow T$  by putting  $g(\gamma) = f[g(\gamma-1)]$ . In this case, obviously, a sequence will be obtained that also satisfies requirements a) and b). (Here we use the fact that  $f(T) \subseteq T$ .)

If  $\gamma$  is an ordinal number of the second kind, then we first show that the closure of the set of points  $\Omega = \{g(\alpha), \alpha < \gamma\}$  is bicomcompact. According to the hypothesis of the theorem and the definition of a generalized condensing operator, for this it is sufficient to show that the set

$$\Omega \setminus \overline{f(\Omega)} \quad (1)$$

bicomcompact. Obviously,  $f(\Omega) = \{g(\alpha), \alpha < \gamma, \alpha \text{ a first-kind ordinal}\}$ , since by the induction hypothesis  $f[g(\alpha)] = g(\alpha+1)$  (whence, incidentally, the inclusion  $f(\Omega) \subseteq \Omega$  follows). Further, if  $\alpha$  is a second-kind ordinal,  $\alpha < \gamma$ ,  $\alpha \neq 0$ , then, by assumption,  $g(\alpha)$  is a generalized limit point of the sequence  $g : A^\alpha \rightarrow T$ , and, consequently,

$$g(\alpha) \in \overline{f(\Omega)}.$$

Thus,

$$\Omega \setminus \{g(0)\} \subseteq \overline{f(\Omega)},$$

but then, a fortiori,

$$\Omega \setminus \{g(0)\} \subseteq \overline{f(\Omega)}.$$

It follows that

$$\Omega \setminus \overline{f(\Omega)} \subseteq \{g(0)\},$$

so that the set (1), and consequently also  $\overline{\Omega}$ , is bicomcompact. Therefore the generalized sequence  $g : A^\gamma \rightarrow T$  has a limit point, which may be taken as  $g(\gamma)$ .

Thus, the existence of a generalized sequence satisfying conditions a) and b) has been proved for every ordinal  $\alpha$ . We construct a transfinite sequence of sets  $\{\Omega_\alpha\}$  by the formulas:

$$\Omega_0 = T; \quad \Omega_\alpha = \overline{f(\Omega_{\alpha-1})}, \text{ if } \alpha \text{ is a first-kind ordinal; } \quad \Omega_\alpha = \bigcap_{\beta < \alpha} \Omega_\beta, \text{ if } \alpha \text{ is a second-kind ordinal.}$$

It is not hard to see that all these sets lie in  $T$ , are convex, closed, and invariant with respect to  $f$ :  $f(\Omega_\alpha) \subseteq \Omega_\alpha$ . It is also obvious that, from some point on, the sequence  $\{\Omega_\alpha\}$  becomes stationary:  $\Omega_{\gamma+1} = \Omega_\gamma$ . This means that  $\overline{\text{co}}f(\Omega_\gamma) = \Omega_\gamma$ , i.e. the set  $\Omega_\gamma \setminus \overline{\text{co}}f(\Omega_\gamma)$  is empty. Hence, from the definition of a generalized condensing operator  $f$ , the bicomcompactness of the set  $\Omega_\gamma$  follows. If we show that  $\Omega_\gamma$  is nonempty, then the proof can be completed by referring to the Schauder-Tikhonov theorem ((3), p. 493), according to which a continuous operator  $f$  that maps a convex bicomcompact set of a locally convex linear topological space into itself has a fixed point in this set.

In order to prove the nonemptiness of  $\Omega_\gamma$ , we consider the generalized sequence  $g : A^\gamma \rightarrow T$ , whose existence was proved above, and verify by induction on  $\alpha$  that  $g(\alpha) \in \Omega_\alpha$  for  $\alpha \leq \gamma$ . Obviously,  $g(0) \in \Omega_0 = T$ . Further, if  $\alpha$  is a first-kind ordinal, then from  $g(\alpha - 1) \in \Omega_{\alpha-1}$  it follows that

$$g(\alpha) = f[g(\alpha - 1)] \in f(\Omega_{\alpha-1}) \subseteq \overline{\text{co}}f(\Omega_{\alpha-1}) = \Omega_\alpha.$$

Let  $\alpha$  be a second-kind ordinal, and suppose that  $g(\beta) \in \Omega_\beta$  for  $\beta < \alpha$ . By b),  $g(\alpha)$  is a generalized limit point of the sequence  $g : A^\alpha \rightarrow T$ . Fix any  $\beta < \alpha$  and show that  $g(\alpha) \in \Omega_\beta$ . From this the inclusion  $g(\alpha) \in \Omega_\alpha$  will immediately follow, since  $\Omega_\alpha = \bigcap_{\beta < \alpha} \Omega_\beta$ . Let  $U$  be an arbitrary neighborhood of the point  $g(\alpha)$ . By the definition of a generalized limit point, there exists  $\beta' \geq \beta$  such that  $g(\beta') \in U$ . Since  $g(\beta') \in \Omega_{\beta'}$ , it follows that  $\Omega_{\beta'} \cdot U \neq \emptyset$ . But then, a fortiori,  $\Omega_\beta \cdot U \neq \emptyset$ . Thus, every neighborhood of the point  $g(\alpha)$  intersects  $\Omega_\beta$ , i.e.  $g(\alpha) \in \overline{\Omega_\beta} = \Omega_\beta$ . The theorem is completely proved.

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## CITED LITERATURE

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*Note: Figure translations are in progress. See original paper for figures.*

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