On the Convergence of Solutions of the Cartwright-Littlewood Equation

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Abstract

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On the Convergence of Solutions of the Cartwright-Littlewood Equation

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Consider the equation

$$\ddot{y} + kf(y)\dot{y} + g(y) = kbp(t), \tag{1}$$

where the functions $f, g, p \in C^2(-\infty, +\infty)$, and k is a large parameter. Let the following assumptions be satisfied.

A. $p(t)$ is 2π-periodic, its mean value is equal to 0; $p(t+\pi) = -p(t)$; $p(t) = 0$ if and only if $t = \pi/2 \pmod{2\pi}$. Denote by $P(t)$ that antiderivative of $p(t)$ whose mean value is 0, and normalize $p(t)$ so that the greatest value of $P(t)$ is equal to 1 and corresponds to $t = \pi/2 \pmod{2\pi}$.

B. $f(y)$ is even, has a unique pair of zeros ± 1, $f'(1) > 0$, and

$$\inf_{y \geq 2} f(y) > 0.$$

Put

$$F(y) = \int_0^y f(y)\,dy, \qquad b_0 = F(-1).$$

By H and H_0 we denote the largest roots of the equations $F(y) = b$ and $F(y) = b_0$, respectively.

C. $g(y)$ is odd and $0 < l_1 < g' < l_2$.

These conditions are satisfied by the equation

$$\ddot{y} + k(y^2 - 1)\dot{y} + y = kb\cos t.$$

In what follows we assume that $b > b_0$, and k is sufficiently large. Everywhere by L we mean a positive constant independent of k and not fixed in the absence of an index, even within the same proposition. $\xi > 0$ always denotes a quantity $O(\exp\{-Lk\})$.

The case $0 < b < b_0$ was considered in detail in (1). There the hypothesis was also put forward that, in the case $b > b_0$, all solutions of equation (1) converge as $t \to +\infty$. Below the validity of this hypothesis is proved.

Theorem. If $b > b_0$, then there exists k_0 such that for all $k > k_0$ and any two solutions $y_1(t)$ and $y_2(t)$ of equation (1),

$$\lim_{t\to+\infty} |y_1(t) - y_2(t)| = 0, \qquad \lim_{t\to+\infty} |\dot{y}_1(t) - \dot{y}_2(t)| = 0.$$

Below a scheme of the proof is given, with some details made more concrete.

Wendell proved (2) that for any $\varepsilon > 0$ one can specify $k_0(\varepsilon)$ and $n_0(\varepsilon)$ such that, for all $k > k_0(\varepsilon)$ and $n > n_0(\varepsilon)$, the inequalities hold (n are natural numbers)

$$|y(\pm\pi/2 + 2n\pi) \mp H| < \varepsilon, \qquad |\dot{y}(\mp\pi/2 + 2n\pi)| < Lk^{1/2}. \tag{2}$$

In what follows all solutions are considered only for

$$t > \pi/2 + 2n_0(\varepsilon)\pi,$$

where ε is sufficiently small. It also follows from the arguments in (2) that there exists $\delta > 0$ such that $|y(t)| > H_0$ for $t \in (Z-3\delta, Z+3\delta)$ and $t \in (Z'-3\delta, Z'+3\delta)$. Here and below $Z = \pi/2 \pmod{2\pi}$, $Z' = -\pi/2 \pmod{2\pi}$, and in all statements formulated further only neighboring Z and Z' are considered.

Using the fact that $p(t) > 0$ for $t \in (Z', Z)$ and $p(t) < 0$ for $t \in (Z, Z')$, it is easy to obtain the following result.

Lemma 1. For every $d > 0$ one can specify an $A > 0$ such that $\dot{y}(t) > A$ for $t \in (Z' + d, Z - d)$ and $\dot{y}(t) < -A$ for $t \in (Z + d, Z' - d)$.

Lemma 1 makes it possible to determine the moments $U, K' \in (Z, Z')$ and $U', K \in (Z', Z)$ by means of the equalities $y(U) = 1$, $y(U') = -1$, $y(K) = \frac{1}{2}(1 + H_0)$, $y(K') = -\frac{1}{2}(1 + H_0)$. Let us also put $\dot{y}(u) = v$, $\dot{y}(U') = v'$. In what follows all results are formulated for one of the semiperiods (Z, Z') or (Z', Z), but they are valid (with obvious modifications) also for the other.

Using (1), it is not difficult to obtain the estimates

$$Lk^{1/3} < v' < Lk^{1/2}; \tag{3}$$

$$v' + k[b_0 - F(y)] \le \dot y(t) \le$$

$$\le v' + k[b_0 - F(y)] + L\big[({v'}^2 + Lk(y+1))^{1/2} - v'\big] \tag{4}$$

for $t \in (U', K)$, and moreover $\operatorname{mes}(U', K) < Lk^{-1/2}$.

Now we can prove the following lemma.

Lemma 2. If $(t_1, t_2) \subset (U', K)$, then

$$L_1 \frac{\dot y(t_2)}{\dot y(t_1)} < \exp\left\{-\int_{t_1}^{t_2} f(y)\,dt\right\} < L_2 \frac{\dot y(t_2)}{\dot y(t_1)}.$$

Proof. Put $\eta = y + 1$. Then from (4) we have

$$v' + kG(\eta) \le \dot y \le v' + kG(\eta) + \Phi(\eta); \tag{5}$$

$$\Phi(\eta) = L\big[({v'}^2 + Lk\eta)^{1/2} - v'\big], \qquad L\eta^2 < G(\eta) = b_0 - F(y) < L\eta^2. \tag{6}$$

Let $\eta_1 = \eta(t_1)$, $\eta_2 = \eta(t_2)$ and $\eta^* = k^{-1/3}$. Then

$$-k\int_{t_1}^{t_2} f(y)\,dt = \int_{\eta_1}^{\eta_2} \frac{k\,d\sigma}{\dot y}. \tag{7}$$

Consider the case $\eta_1 \le \eta^* \le \eta_2$. The estimates below follow from (3), (5), (6). From (5) we have

$$v' \le \dot y(t_1) < Lv'. \tag{8}$$

Further, by virtue of (7),

$$-k\int_{t_1}^{t_2} f(y)\,dt \le \int_0^{\eta^*} \frac{k\,dG}{v'} + \int_{\eta^*}^{\eta_2} \frac{k\,dG}{v' + kG},$$

whence

$$-k\int_{t_1}^{t_2} f(y)\,dt \le \frac{kG(\eta^*)}{v'} + \ln\frac{v' + kG(\eta_2)}{v' + kG(\eta^*)} < \ln\frac{L\dot y(t_2)}{v'}. \tag{9}$$

Put

$$I = \int_{\eta^*}^{\eta_2} \frac{d\Phi}{v' + kG + \Phi}.$$

By virtue of (7),

$$-k \int_{t_1}^{t_2} f(y)\,dt + I > \int_{\eta^*}^{\eta_2} \frac{d(v' + kG + \Phi)}{v' + kG + \Phi}.$$

A direct calculation shows that $I < L$. Then

$$-k \int_{t_1}^{t_2} f(y)\,dt > \ln \frac{v' + kG(\eta_2) + \Phi(\eta_2)}{v' + kG(\eta^*) + \Phi(\eta^*)} - L > \ln \frac{L\dot{y}(t_2)}{v'},$$

Hence, and from (9),

$$\frac{L\dot{y}(t_2)}{v'} < \exp\left\{ -\int_{t_1}^{t_2} f(y)\,dt \right\} < \frac{L\dot{y}(t_2)}{v'},$$

and the assertion of the lemma follows from (8). Finally, the validity of the lemma when $\eta_1, \eta_2 \leqslant \eta^*$ and $\eta_1, \eta_2 \geqslant \eta^*$ follows from the case considered.

Let us now consider two arbitrary solutions $y_1(t)$ and $y_2(t)$ of equation (1). If X is a function of the solutions, then put $\Delta X = X(y_2) - X(y_1)$. Further, let

$$w = y_2 - y_1, \qquad u = \Delta F / w, \qquad \gamma = \Delta g / w \quad (l_1 < \gamma < l_2),$$

$$T = k \int_{Z'}^{t} u\,dt.$$

By the indices 1 and 2 we denote quantities connected with $y_1(t)$ or $y_2(t)$. Finally, put $c(t) = \dot{w} - \dot{T}w$. From (1) we obtain

$$c(t_2) - c(t_1) = -\int_{t_1}^{t_2} \gamma w\,dt. \tag{10}$$

The following two results are basic for what follows.

Lemma 3. There exists ζ^* with the following property: if $\zeta^* < w(Z) < k^{-11}$, then $0 < w(t) < k^{-5}$ for $t \in (Z, U_2')$, and the ratio of any two of the quantities $kw(Z), kw(Z'), c(Z), c(Z'), c(U_1), c(U_2')$ lies between two L' s.

Lemma 4. For any ζ', if $|w(Z)| < \zeta'$, then $|w(t)|, |\dot{w}(t)|, |c(t)| < \zeta(\zeta')$ for $t \in (Z, U')$, where $U' = \min\{U_1', U_2'\}$.

Lemmas 3 and 4 are analogues of Lemmas 24 and 25 of (^{1}), with the difference that in (^{1}) these lemmas are valid only for solutions belonging to a certain class. For solutions of this class, a lemma (Lemma 12 in (^{1})) analogous to Lemma 2 is valid, but its proof is based on facts which do not hold in the case under consideration. Using Lemma 2, and also Lemma 1 and the inequality $\operatorname{mes}(U', K) < Lk^{-1/2}$, we may, in proving Lemmas 3 and 4, follow (^{1}), the arguments even being simplified; we omit the proofs and refer the reader directly to (^{1}) (Lemmas 13-25).

Lemma 5. If $w(Z) > \zeta^*$, then $w(t) > Lk^{-1}c(U_1)$ for $t \in (Z' + \delta, Z' + 2\delta)$.

Proof. Up to the first intersection after Z' of the trajectories $y_1(t)$ and $y_2(t)$ in the yt-plane, $w(t) > 0$, and, consequently, the inequality (^{1}), p. 55, is valid:

$$w(t) \int_{Z'}^{t} e^{T}\, dt \geq c(Z')\varphi(t) \left[\int_{Z'}^{t} e^{T}\, dt - \frac{w(Z')}{c(Z')}\psi(t) - \chi(t) \right], \tag{11}$$

where

$$\varphi(t) = e^{-T} \int_{Z'}^{t} e^{T}\, dt, \qquad \psi(t) = l_2 \iint_{Z' \leq \xi \leq \eta \leq T} \exp\{T(\eta) - T(\xi)\}\, d\xi\, d\eta,$$

$$\chi(t) = l_2 \iiint_{Z' \leq \xi \leq \eta \leq \zeta \leq t} \exp\{T(\xi) - T(\eta) + T(\zeta)\}\, d\xi\, d\eta\, d\zeta.$$

Put now

$$\tau(t) = k \int_{Z'}^{t} f(y_1)\, dt.$$

Since $f(y_1) > L$ for $t \in (Z', Z' + 2\delta)$, it follows that

$$\tau \geq Lk(t - Z').$$

Therefore, for $t \in (Z' + \delta, Z' + 2\delta)$,

$$\int_{Z'}^{t} e^{-\tau}\, dt < \int_{Z'}^{t} e^{-Lk(t-Z')}\, dt < Lk^{-1}; \tag{12}$$

$$Lk^{-1} < e^{-\tau} \int_{Z'}^{t} e^{\tau}\, dt < Lk^{-1}. \tag{13}$$

$$\iint_{Z' < \xi < \eta < t} \exp\{\tau(\xi) - \tau(\eta)\}\, d\xi d\eta = \int_{Z'}^{t} d\eta \left(e^{-\tau(\eta)} \int_{Z'}^{\eta} e^{\tau(\xi)}\, d\xi \right) < Lk^{-1}. \tag{14}$$

By Lemma 3, $0 < w(t) < k^{-5}$ for $t \in (Z' + \delta, Z' + 2\delta)$, and therefore $\dot{\tau} = T + O(k^{-4})$ and $\exp\{\pm T\} = \exp\{\pm\tau\}(1 + O(k^{-4}))$. Hence it follows that the estimates (12)–(14) remain valid if τ is replaced in them by T. Then

$$\psi < L\left(\int_{Z'}^{t} e^{T(\eta)}\,d\eta\right)\left(\int_{Z'}^{t} e^{-T(\xi)}\,d\xi\right) < Lk^{-1}\int_{Z'}^{t} e^{T}\,dt,$$

$$\chi < L\left(\int_{Z'}^{t} e^{T(\zeta)}\,d\zeta\right)\left(\iint_{Z'<\xi<\eta<t} \exp\{T(\xi) - T(\eta)\}\,d\xi d\eta\right) < Lk^{-1}\int_{Z'}^{t} e^{T}\,dt$$

by virtue of (12) and (14). Hence, from Lemma 3 and from (11) and (13), we obtain, for $t \in (Z' + \delta, Z' + 2\delta)$,

$$w(t) > Lk^{-1}c(Z') > Lk^{-1}c(U_1).$$

Lemma 6. If $|w(Z)| < k^{-11}$, then

$$|c(U_2')| < (1 - Lk^{-1})|c(U_1)| + \zeta.$$

Proof. If $|w(Z)| \leq \xi^*$, then the assertion of the lemma is trivial in view of Lemma 4. Let now $w(Z) > \xi^*$. Then $w(t) > 0$ for $Z \leq t \leq U_2'$, and by Lemma 3 and from (10) we have

$$c(U_2') - c(U_1) < -L\int_{U_1}^{U_2'} w\,dt.$$

Hence, by Lemma 5,

$$c(U_2') - c(U_1) < -Lk^{-1}c(U_1),$$

which proves the lemma.

Proof of the theorem. By Lemma 6, for any two solutions of equation (1) for which $|w(Z)| < k^{-11}$, either $c(U_1), c(U_2') = O(\zeta)$, or $c(U_1), c(U_2')$ decrease in absolute value (though slowly for large k) and eventually become quantities of order ζ. Then, by Lemma 4, $w(t), \dot{w}(t) = O(\zeta)$ for all sufficiently large t. This proves convergence to within ζ. The proof of true convergence is the same as in (1) (§§ 55-56). As a result we obtain convergence under the condition $|w(Z)| < k^{-11}$. But from (2) it follows that then $\lim_{t\to+\infty} w(t) = 0$ for any two solutions of equation (1), and $\lim_{t\to+\infty} \dot{w}(t) = 0$, since $\dot{w}(t)$ is bounded as $t \to +\infty$.

The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.

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