AUTOMATON MAPPINGS AND MATRICES OVER THE SYMMETRIC SEMIGROUP OF AN ABELIAN GROUP

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Abstract

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AUTOMATON MAPPINGS AND MATRICES OVER THE SYMMETRIC SEMIGROUP OF AN ABELIAN GROUP

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In this note we set forth results of an investigation of automaton mappings admitting representations by matrices over the symmetric semigroup of an alphabet in which the structure of an additive semigroup with zero is defined or, in particular, the structure of an abelian group; a special case of such mappings are those representable by matrices over a ring with identity or an almost field defined in the alphabet, including linear automaton mappings over a commutative ring with identity, studied by Nērōde (see ([1])). These results were obtained by applying the theorems and methods of the author' s paper ([2]), to which the present note is adjacent.

1. Let the alphabet X be endowed with the structure of an additive semigroup with zero 0, which we shall denote by $(X, +)$. Addition in X naturally induces addition in $S'(X)$, with respect to which the latter set forms a semigroup $(S'(X), +)$ with zero ζ_0 (in general, ζ_a for $a \in X$ denotes the mapping from $S'(X)$ sending any element $x \in X$ to a).

Consider square matrices of order k $(k = 0, 1, \dots, \infty)$ over $S'(X)$, and in the case $k = \infty$ impose the condition that the rows be finite. Since in $S'(X)$ two operations are defined—addition and multiplication—addition and multiplication of matrices of the same order are defined in the usual way, and the definition of multiplication is correct in view of the finiteness of the rows. A matrix over $S'(X)$ all of whose elements are endomorphisms of the semigroup $(X, +)$ will be called endomorphic; such matrices, in the case when $(X, +)$ is an abelian group, were studied by Z. M. Kishkina (see ([3,4])) in connection with the problem of describing the endomorphism ring of a direct sum of abelian groups.

To a matrix (φ_{ij}) of order k over $S'(X)$ there corresponds a mapping φ of the semigroup $(X^k, +)$ into itself if, for $x = (x_0, \dots, x_{k-1})$, we put

$$\varphi x = x' = (x'_0, \dots, x'_{k-1}), \quad \text{where} \quad x'_i = \sum_{j=0}^{k-1} \varphi_{ij} x_j \quad (i = 0, 1, \dots, k-1).$$

This definition is correct in connection with the condition that the rows be finite.

The operation of addition in X defines a function $a(x)$, given on X with values in $S'(X)$, which for any $y \in X$ is specified by the equality $a(x)y = x + y$. By means of this function, to each infinite triangular matrix (φ_{ij}) $(i, j = 0, 1, \dots;\ \varphi_{ij} = \zeta_0$ for $j > i)$ over $S'(X)$ we associate the functional semimatrix $(a_{ij}(x))_{j \le i} = (a(\varphi_{ij} x))_{j \le i}$ (see (2)).

Proposition 1. *An infinite triangular matrix (φ_{ij}) over $S'(X)$ and the corresponding semimatrix $(a(\varphi_{ij} x))_{j \le i}$ induce one and the same automaton mapping.*

We shall call a triangular infinite matrix (ψ_{ij}) over $(S'(X), +)$ the derivative with respect to the vector $\mathbf{a} \in X^s$ $(s < \infty)$ of a triangular infinite matrix (φ_{ij}) over $(S'(X), +)$, if

$$\psi_{ij} = \begin{cases} \zeta_{\varphi_{(s+i)0} a_0} + \zeta_{\varphi_{(s+i)1} a_1} + \dots + \zeta_{\varphi_{(s+i)(s-1)} a_{s-1}} + \varphi_{(s+i)s}, & \text{if } j = 0; \\ \varphi_{(s+i)(s+j)}, & \text{if } j > 0; \end{cases}$$

we shall denote it by $(\varphi_{ij})'_a$.

An infinite triangular matrix (φ_{ij}) over $(S'(X), +)$ will be called periodic if the submatrix $(\varphi_{ij})_{j \le i}$ lying below its main diagonal is periodic (see (2)).

Theorem 1. *Let an automaton mapping α in the alphabet X, under some structure $(X, +)$ of an additive semigroup with zero 0 in X, be given by an infinite triangular matrix (φ_{ij}) over $S'(X)$. Construct an automaton whose states are the matrices $(\varphi_{ij})'_a$, and whose transition and output functions are given by the formulas*

$$\delta[(\varphi_{ij})'_a, x] = (\varphi_{ij})'_{ax}, \qquad \lambda[(\varphi_{ij})'_a, x] = (\varphi_i)'_a x;$$

as the initial state we choose (φ_{ij}).

The automaton constructed is the minimal automaton inducing the mapping α.

If the matrix (φ_{ij}) is periodic, then the mapping is finite-automaton, and the constructed minimal automaton has no more than $(k+m)n^{\chi+\mu}$ states, where k, m are, respectively, the numbers of elements before the period and in the period for the sequence of columns of the submatrix $(\varphi_{ij})_{j \le i}$; χ_j, μ_j are the same numbers for its j-th column, $\chi = \max(\chi_0, \dots, \chi_{k+m-1})$, $\mu = \text{l. c. m.}(\mu_0, \dots, \mu_{k+m-1})$.

The set of automaton mappings induced, for a given semigroup $(X, +)$ with zero, by infinite triangular matrices is closed.

Proof is obtained by passing from (φ_{ij}) to the corresponding functional submatrix $(a(\varphi_{ij}x))_{j\leq i}$ and applying Theorem 1 of [2].

Theorem 2. *In order that an automaton mapping α in the alphabet X be induced by an infinite triangular matrix (φ_{ij}) over $(S'(X), +)$, under some structure $(X, +)$ of a semigroup in X with zero 0, it is necessary and sufficient that the functions of its representation*

$$\alpha = [\alpha_0, \alpha_1(x_0), \dots, \alpha_r(x_0, \dots, x_{r-1}), \dots] \tag{1}$$

as an element of the wreath product of the symmetric semigroups $S'(X)$ admit decompositions of the form

$$\alpha_r(x_0, \dots, x_{r-1}) = \alpha(\varphi_{r0}x_0) \dots \alpha(\varphi_{r-1}x_{r-1})\varphi_{rr}$$

$(r = 0, 1, \dots)$ with some function $\alpha(x)$ satisfying the conditions:
1) $\alpha(0)y = y$, $\alpha(x)0 = x$ for arbitrary $x, y \in X$, and 2) $\alpha(\varphi y) = \varphi\alpha(y)$ for arbitrary $\varphi \in \alpha(X)$ and $y \in X$.*

2. Let now $(X, +)$ be an Abelian group. Then both $(S'(X), +)$ and $(X^k, +)$ are Abelian groups. A matrix (φ_{ij}) over $(S'(X), +)$ will be called homogeneous if, for all i, j, $\varphi_{ij}0 = 0$. A matrix (φ_{ij}) will be called equivalent to zero if all its elements are constant functions and the sum of the elements of each row is equal to the zero ξ_0 of the group $(S'(X), +)$.

Proposition 2. *Matrices of order k equivalent to zero form a subgroup in the additive group of all matrices of order k over $(S'(X), +)$.*

We shall call two matrices of order k equivalent if they belong to the same coset with respect to this subgroup.

Proposition 3. *Two matrices of order k are equivalent if and only if they induce one and the same mapping of the set X^k into itself.*

Proposition 4. *Two equivalent homogeneous matrices are equal.*

We shall agree to say that a matrix (φ_{ij}) has normal form if all its off-diagonal elements are homogeneous in the sense that they leave 0 fixed.

Proposition 5. *In each equivalence class of matrices of order k*

of order over $(S'(X), +)$, there exists, moreover, a unique matrix of normal form.

We shall represent mappings that admit specification by matrices over $(S'(X), +)$ exclusively by matrices of normal form. We restrict ourselves to infinite triangular matrices; the mappings of the set X^∞ into itself induced by them are automaton mappings.

Theorem 3. For an automaton mapping in the alphabet X with the structure of an Abelian group $(X, +)$, induced by a triangular matrix of infinite order over $(S'(X), +)$, to be finite-automaton, it is necessary and sufficient that the matrix of normal form defining it be periodic.

3. Let the alphabet X be endowed with the structure $(X, +, \cdot)$ of a ring with identity or of a near-field (see (5)). A matrix (a_{ij}) over $(X, +, \cdot)$ of k-th order defines, in the usual way, a mapping of the set X^k into itself. The same mapping is induced by the matrix $(\mu_{a_{ij}})$ over $S'(X)$, where μ_a denotes the mapping $x \to ax$ $(a, x \in X)$ of the set X into itself. This makes it possible to solve the problem of synthesizing a minimal automaton for an automaton mapping given by an infinite triangular matrix over $(X, +, \cdot)$ on the basis of Theorem 1. In addition, it follows from this that

Theorem 4. For an automaton mapping α in the alphabet X with the structure $(X, +, \cdot)$ of a ring with identity or of a near-field, specified by an infinite triangular matrix (a_{ij}) over $(X, +, \cdot)$, to be finite-automaton, it is necessary and sufficient that the matrix (a_{ij}) be periodic. In this case the number of states of the minimal automaton inducing α does not exceed $(k + m)n^{\chi+\mu}$, where the meanings of the notations k, m, χ, μ are the same as in Theorem 1.

From Theorem 4 one obtains, in particular, results for linear automaton mappings in the alphabet X with the structure $(X, +, \cdot)$ of a commutative ring with identity; and from them, by means of the transformation $(a_{ij}) \to (a_{i-j,j})$, one can obtain the criterion for finite-automaton-ness of an automaton linear mapping obtained by Nerode (see (1)).

4. Consider the binary alphabet $X = \langle x, y\rangle$. There are only two structures of Abelian (cyclic) groups of the second order in X, and they are completely determined by which of the two elements x and y is to be taken as zero 0. We give necessary and sufficient conditions, in the language of regular expressions (see (6)), for the representability of a finite-automaton mapping by a matrix over $(S'(X), +)$ for at least one of these group structures $(X, +)$.

Let $\mathbf{i} = (i_0, \dots, i_{k-1})$ be a k-dimensional $(k < \infty)$ vector with components from the set $\langle 0, 1\rangle$. We shall call it even if an even number of its components are equal to 1, and odd otherwise. We denote the set of even k-dimensional vectors by O_k, and the set of odd ones by I_k. A product of the form $A_{i_0}^{(0)} A_{i_1}^{(1)} \cdots A_{i_{k-1}}^{(k-1)}$ of subsets of the alphabet X, where $i_0, i_1, \dots, i_{k-1} = 0, 1$, will be denoted by $A_{\mathbf{i}}^{[0,k-1]}$; moreover, below always, if $A_0 \subseteq X$, then $A_1 = X \setminus A_0$.

A regular expression W will be called $(0, x)$-normal if it has the following structure:

$$1)\ W = P \vee_{r+q-1} R; \qquad 2)\ P = \bigvee_{k=0}^{r} P_k\ (q, r < \infty);$$

3) For each $k = 0, 1, \dots, r + q - 1$ there exists a sequence of subsets $P_0^{(k,0)}, P_0^{(k,1)}, \dots, P_0^{(k,k)}$ of the alphabet X such that

$$P = \bigvee_{\mathbf{i} \in O_{k+1}} P_{i_0}^{(k,0)} P_{i_1}^{(k,1)} \cdots P_{i_k}^{(k,k)} \quad (k = 0, 1, \dots, r+q-1);$$

$$4)\ \text{If } k \neq l, \text{ then } x \in P_0^{(k,l)}; \qquad 5)\ R = \bigvee_{i=0}^{s-1} R_i \ (s < \infty);$$

6) There exist sequences $A_0^{(i,0)}, \dots, A_0^{(i,r-0)}, B_0^{(i,0)}, \dots,$ $\dots, B_0^{(i,s-1)}; C_0^{(i,0)}, \dots, C_0^{(i,q-1)}$ of subsets of the alphabet X, such that

$$R_i = \left(\bigvee_{ikl \in O_{r+i+q+1}} A_i^{(i,[0,r-1])} S_i B_k^{(i,[0,i])} C_1^{(i,[0,q-1])} \right) \vee$$

$$\vee \left(\bigvee_{ikl \in I_{r+i+q+1}} A_i^{(i,[0,r-1])} T_i B_k^{(i,[0,i])} C_1^{(i,[0,q-1])} \right);$$

7) $x \in A_0^{(i,j_1)}, B_0^{(i,j_2)}, C_0^{(i,j_3)} (j_1 = 0, \dots, r-1;\ j_2 = 0, \dots, s-1;\ j_3 = 0, \dots, q-1);$

8) $S_i = \{B_0^i \vee B_I^i \{B_0^i\} B_I^i\}; \quad T_i = \{B_0^i\} \cdot B_I^i \{B_0^i \vee B_I^i \{B_0^i\} B_I^i\};$

9) $B_0^i = \bigvee_{k \in O_s} B_k^{(i,[0,s-1])}, \quad B_I^i = \bigvee_{k \in I_s} B_k^{(i,[0,s-1])}.$

A regular event in the alphabet X will be called $(0, x)$-normal if among the regular expressions representing it there is a $(0, x)$-normal expression.

Theorem 5. In order that a finite-automaton mapping α in the alphabet $X = \langle x, y \rangle$, specified by the canonical system of events $\langle W_x, W_y \rangle$, be induced by a matrix over $(S'(X), +)$ under such a group structure in X, when the letter x serves as the zero 0, it is necessary and sufficient that the event W_x be $(0, x)$-normal. If, moreover, (φ_{ij}) is the matrix for α, $\vec{\alpha}_i = (\alpha_0^i, \dots, \alpha_{r-1}^i)$, $\vec{\beta}_i = (\beta_0^i, \dots, \beta_{s-1}^i)$, $\vec{\gamma}_i = (\gamma_0^i, \dots, \gamma_{q-1}^i)$ are vectors determining it by the lemma on periodic semimatrices (2), then the equalities

$$P_0^{(k,j)} = P_x^{\varphi_{kj}}, \quad A_0^{(i,j_1)} = P_x^{{\alpha_{j_1}}^i}, \quad B_0^{(i,j_2)} = P_x^{{\beta_{j_2}}^i},$$

$$C_0^{(i,j_3)} = P_x^{{\gamma_{j_3}}^i}$$

hold

$$(j \leq k;\ j, k = 0, 1, \dots, r+q-1;\ i = 0, 1, \dots, s-1;\ j_1 = 0, \dots, r-1;$$
$$j_2 = 0, \dots, s-1;\ j_3 = 0, \dots, q-1),$$

where P_x^φ for $\varphi \in S'(X)$ denotes the complete inverse image of the letter x under the mapping φ. This makes it possible to pass from specifying α by a matrix to specifying it by regular $(0, x)$-normal expressions, and conversely.

Let a $(0, x)$-normal expression W be given. Construct a new expression $\overline{W}$, satisfying conditions 1), 2), 3*), 4), 5), 6*), 7)–9), where 3*) is obtained from 3) by replacing O_{k+1} by I_{k+1}, and 6*) is obtained from 6) by replacing S_i by T_i and conversely. Such an expression will be called $(1, y)$-normal, and the event represented by it a $(1, y)$-normal event conjugate to the $(0, x)$-normal event W.

Theorem 5*. In order that a finite-automaton mapping α in the alphabet $X = \langle x, y\rangle$, specified by the canonical system of events $\langle W_x, W_y\rangle$, be matrix under such a group structure in X, when $x = 0$, it is necessary and sufficient that the event W_y be $(1, y)$-normal, conjugate to the $(0, x)$-normal event W_x.

Theorem 6. In order that a finite-automaton mapping α in the alphabet $X = \langle x, y\rangle$, specified by the canonical system of regular expressions $\langle W_x, W_y\rangle$, be matrix under some group structure in X, it is necessary and sufficient that W_x be either $(0, x)$-normal or $(1, x)$-normal, or, equivalently, that W_y be either $(1, y)$-normal or $(0, y)$-normal, respectively.

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Note: Figure translations are in progress. See original paper for figures.

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