A CYCLIC TWO-DIMENSIONAL COMPACTUM CONTAINING NO IRREDUCIBLY CYCLIC TWO-DIMENSIONAL SUBCOMPACTUM

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Abstract

Full Text

MATHEMATICS

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A CYCLIC TWO-DIMENSIONAL COMPACTUM CONTAINING NO IRREDUCIBLY CYCLIC TWO-DIMENSIONAL SUBCOMPACTUM

(Presented by Academician A. N. Tikhonov on 19 III 1968)

Definition 1. An n-dimensional compactum F is called **cyclic** if there exists an abelian group $\mathfrak{A}$ and an n-dimensional true cycle (2) with coefficients in $\mathfrak{A}$, which is not homologous to zero in F (1).

Definition 2. An n-dimensional cyclic compactum F is called **irreducibly cyclic** (or an n-dimensional closed Cantor manifold (1)) if F contains no n-dimensional cyclic proper subcompactum $F' \subset F$.

Remark. If an n-dimensional compactum F is cyclic, then there exists in F some n-dimensional convergent cycle (2) with coefficients in $\mathfrak{R}_1$, which is not homologous to zero (the symbol $\mathfrak{R}_1$ here denotes the additive group of all rational numbers reduced modulo 1). This fact is an almost immediate consequence of the so-called convergence theorem (2).

The present note solves the following problem posed by P. S. Alexandrov (Problem III in (1), p. 227):

Does every n-dimensional cyclic compactum contain some n-dimensional closed Cantor manifold?

We shall show that the answer is negative even for $n = 2$.

First take the plane disk

$$K = \{(x, y) : x^2 + y^2 \leq 1\}$$

and the sequence of circles lying in this disk,

$$L_k = \left\{(x, y) : x^2 + y^2 = \frac{1}{k}\right\}$$

$(k = 1, 2, ...)$. Let $\{p_k\}$ be the increasing sequence of all prime numbers.

We now introduce an equivalence relation σ between points of the disk K, defining it as follows: for $x \neq y$, $x\sigma y$ if and only if there exists such a k that x and y are vertices of some regular p_k-gon inscribed in the circle L_k; moreover, of course, $x\sigma x$ for all $x \in K$. Obviously, the decomposition of the disk K into the equivalence classes of σ is lower semicontinuous. Denote by F the quotient space K/σ. Obviously,

$$\dim F = 2.$$

We shall now show that the compactum F constructed above has the following property:

The two-dimensional subcompactum $F' \subseteq F$ is cyclic if and only if it is a neighborhood of the point $[O]$ (where $O = (0,0) \in K$).

Proof. Let $\pi : K \to F$ denote the canonical projection $\pi(x) = [x]$. The set $F' \subseteq F$ is a neighborhood of the point $[O]$ if and only if $\pi^{-1}(F')$ is a neighborhood of the point O.

1°. Suppose that F' is a neighborhood of the point $[O]$. Then there exists a natural number m such that the disk

$$K_m = \left\{(x, y) : x^2 + y^2 \leq \frac{1}{m}\right\}$$

lies entirely in $\pi^{-1}(F')$. To verify the cyclicity of the set F', it suffices to show that the compactum $\pi(K_m)$ is cyclic. Consider the two-dimensional polyhedron P_m, which is constructed as follows: on the disk K_m, we identify all vertices of any regular p_m-gon inscribed in the circ–

ness L_m; in other words, we "wind" the boundary circle of the disk K_m p_m times onto itself. It is obvious that the two-dimensional homology group of the polyhedron P_m with coefficients J_{p_m} (J_{p_m} is the group of residues modulo p_m) is isomorphic to the group J_{p_m}.

Let us now note that, by identifying some points in P_m, we can obtain the compactum $\pi(K_m)$; more precisely: by identifying in K_m the points that are identified by the mapping $\pi|K_m$, but starting with the points lying on L_m, we first obtain a mapping $K_m \to P_m$, and then $\pi_1 : P_m \to \pi(K_m)$.

On the other hand, by identifying into one all points in $\pi(K_m)$ that lie in $\pi(K_{m+1})$, we obtain a mapping $\pi_2 : \pi(K_m) \to P_m$ such that the composition $\pi_2 \circ \pi_1 : P_m \to P_m$ is homotopic to the identity mapping. Consequently, the induced homomorphism

$$(\pi_2 \circ \pi_1)_* = \pi_{2*} \circ \pi_{1*}$$

is the identity of the (nontrivial) group $H_2(P_m, J_{p_m})$; in particular, the homomorphism

$$\pi_{1*} : H_2(P_m, J_{p_m}) \to H_2(\pi(K_m), J_{p_m})$$

is a monomorphism. It follows that the group $H_2(\pi(K_m), J_{p_m})$ is nontrivial. Thus it has been proved that every compact neighborhood of the point $[O]$ in F is cyclic.

Fig. 1

Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

2°. Suppose now that a closed two-dimensional set $F' \subset F$ is not a neighborhood of the point $[O]$. In order to make sure that the compactum F' is not cyclic, it is enough to show that for every $\varepsilon > 0$ there exists an ε-shift of the compactum F' into some noncyclic two-dimensional polyhedron. Consider the polyhedron $\pi(Q_i)$, where

$$Q_i = \left\{ (x, y) : \left(\frac{1}{i+1} \right)^2 \leq x^2 + y^2 \leq 1 \right\} \subset K, \quad i = 1, 2, \ldots$$

First, let us show that $\pi(Q_i)$ is not cyclic. Consider some triangulation T of the polyhedron $\pi(Q_i)$ and an arbitrary abelian group $\mathfrak{A}$. Take a triangulation T' of the polyhedron Q_i such that the mapping π is simplicial. It is obvious that every simplex of the triangulation T' lies entirely in one of the sets

$$M_k = \left\{ (x, y) : \left(\frac{1}{k+1} \right)^2 \leq x^2 + y^2 \leq \left(\frac{1}{k} \right)^2 \right\}, \quad k = 1, 2, \ldots, i,$$

and therefore the mapping π does not identify any two two-dimensional simplices of T'. Let $\alpha : T \to \mathfrak{A}$ be an arbitrary two-dimensional cycle lying in $\pi(Q_i)$, with coefficients in $\mathfrak{A}$. Then $\alpha(\Delta_1) = \alpha(\Delta_2)$ if and only if the two-dimensional simplices Δ_1 and Δ_2 lie in the same $\pi(M_k)$. Let a_k be the value that the cycle α takes on all two-dimensional simplices lying in $\pi(M_k)$. Since $\partial\alpha = 0$, we obtain the equalities:

$$2a_1 = 0, \quad 3a_1 = 3a_2, \quad 5a_2 = 5a_3, \ldots, \quad p_i a_{i-1} = p_i a_i, \quad p_{i+1} a_i = 0.$$

From these equalities it follows that $a_1 = a_2 = \cdots = a_i = 0$, since all p_k are pairwise relatively prime. Hence, in $\pi(Q_i)$ there is not a single nontrivial two-dimensional cycle.

Fig. 2

Second, let us show that for every $\varepsilon > 0$ there exists an ε-shift of the compactum F' into some $\pi(Q_i)$. Thus, let $\varepsilon > 0$ be given. For sufficiently

of sufficiently large index i, the diameter of the set $\pi(K_i)$ is less than ε. Since the set $\pi^{-1}(F')$ is compact and is not a neighborhood of the point O, there exists a point $x_0 \in K_i$ and a neighborhood U of it that does not meet the set $\pi^{-1}(F')$. We may assume that U is a sufficiently small disk meeting none of the circles L_k (see Fig. 1). We may imagine that Fig. 1 represents the space F. We now carry out a mapping $f_1 : F - \pi(U) \to f_1(F - \pi(U))$, "stretching" the hole $\pi(U)$ (as in Fig. 2), but identical outside the set $\pi(K_i)$. The point x_1 (Fig. 2) separates the set $f_1(F - \pi(U))$ into two open sets U_1 and U_2; let $[O] \in U_1$. Define a continuous mapping $f_2 : f_1(F - \pi(U)) \to F - U_1$ by the formula

$$f_2(x) = \begin{cases} x_1 & \text{for all } x \in U_1, \\ x & \text{for all } x \notin U_2. \end{cases}$$

The composition $f_2 \circ f_1$ is an ε-shift, since the points lying in $\pi(K_i)$ do not leave $\pi(K_i)$, while the remaining points remain in their places; this ε-shift moves the set F' ($F' \subseteq F - \pi(U)$) into some $\pi(Q_m)$, since $F - U_1$ is contained in some $\pi(Q_m)$.

Corollary. The compactum F contains no two-dimensional closed Cantor manifold, although it is itself two-dimensional and cyclic.

Remark. It can be shown that if an n-dimensional cyclic compactum F is an ANR space, then it contains some closed n-dimensional Cantor manifold; more precisely, under the conditions $F \in ANR$, $\dim F = n$, every carrier $F' \subseteq F$ of an n-dimensional cycle non-homologous to zero contains a closed n-dimensional Cantor manifold.

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Note: Figure translations are in progress. See original paper for figures.