

# A BOUNDARY-VALUE PROBLEM FOR A SYSTEM OF EQUATIONS OF A BAROTROPIC ATMOSPHERE

1968

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**Abstract**

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UDC 551.509.313

**GEOPHYSICS**

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## **A BOUNDARY-VALUE PROBLEM FOR A SYSTEM OF EQUATIONS OF A BAROTROPIC ATMOSPHERE**

*(Presented by Academician A. A. Dorodnitsyn on 17 VI 1967)*

1. The success that has been achieved in forecasting the motion of the Earth's atmosphere on the basis of geostrophic models was determined, not least of all, by the fact that it proved possible to reduce the system of complete equations describing the evolution of the wind and geopotential  $\Phi$  to a single elliptic-type equation for  $\partial\Phi/\partial t$ . Since in practice one has to deal with a bounded part of the atmosphere, some boundary condition is needed on the lateral boundary, for example  $\partial^2\Phi/\partial t^2 = 0$ . This condition, naturally, introduces a distortion into the "true" motion inside the forecast region, but the character of the solution of the mathematical problem is such that the influence of the boundary condition rapidly decays inward into the region.

Another essential circumstance that made it possible to use geostrophic models in operational practice is that, in numerical integration of the equations by explicit schemes with a step in the spatial coordinates of 300–400 km and with a time step of the order of an hour, the computation turns out to be stable. This makes it possible, on computers of the M-20 type, to obtain a forecast of the geopotential over a sufficiently large territory in a few minutes.

The rejection of geostrophy and a return to the original (primitive) equations was undertaken in the hope of describing the motion of the atmosphere more accurately. However, various mathematical difficulties arise along this path. The simplest analysis shows <sup>(1)</sup> that, in numerical integration of the primitive equations by explicit schemes, a sharp reduction of the time step is required, which considerably increases the forecast computation time. In the works of I. A. Kibel' <sup>(2)</sup> it was indicated how to circumvent this difficulty. It was proposed to take explicitly into account only the nonlinear terms. This makes it possible, without loss of computational stability, to increase the time step.

It is considerably more difficult than in geostrophic models, when integrating the primitive equations, to take into account the influence of motions outside the forecast region on the motions inside it. As boundary conditions on the lateral

surface that do not strongly distort the motion inside the region, one usually chooses conditions of periodicity of the solution or the vanishing of the normal component of the velocity. However, in order that such boundary conditions in fact not strongly distort the character of the motion, it is necessary to enlarge the forecast region enormously.

Another method that makes it possible to increase the time step without loss of computational stability is the use of implicit schemes. But for such schemes the choice of the boundary condition on the lateral boundary becomes especially important; this condition determines whether it is possible to construct a stable difference scheme approximating the differential equation. An example of boundary conditions of this kind is the prescription of the wind velocity at those parts of the boundary where inflow occurs. Such a condition was used by G. I. Marchuk <sup>(3)</sup>.

For a correct account of the influence of lateral boundaries on the behavior of solutions of the primitive equations, the proper formulation of the boundary-value problem and the proof of an existence and uniqueness theorem for the solution are of great importance. In the general case, unfortunately, no such theorems exist. Therefore it is essential to consider a simplified model of the atmosphere, for which such a boundary-value problem can be formulated.

In the present article a boundary-value problem is posed for a system of equations describing motion in a barotropic atmosphere, and it is shown that, using the method proposed by V. I. Yudovich, one can prove the existence and uniqueness of its solution.

§2. Plane motion in a barotropic atmosphere can be described by the following system of equations (7):

$$\partial \mathbf{V} / \partial t + (\mathbf{V} \cdot \nabla) \mathbf{V} + \nabla \Phi = \mathbf{F}, \quad (1)$$

$$\nabla \cdot \mathbf{V} = 0, \quad (2)$$

where  $\mathbf{V} = \{u, v, 0\}$  is the velocity vector;  $\mathbf{F} = \mathbf{V} \times \mathbf{l}$ ;  $\mathbf{l} = \{0, 0, l\}$ ;  $l$  is the Coriolis parameter. We shall seek the solution of the system (1)–(2) inside a simply connected domain  $\Omega$  with boundary  $S$ . Introduce the vector  $\mathbf{n} = \{\cos(\widehat{nx}), \cos(\widehat{ny}), 0\}$ , which is the outward normal to the boundary  $S$ . To the system we adjoin the following boundary and initial conditions:

$$\mathbf{V} \cdot \mathbf{n} \Big|_S = \gamma(\mathbf{x}, t), \quad (3)$$

$$\mathbf{V}(\mathbf{x}, 0) = \mathbf{a}(\mathbf{x}). \quad (4)$$

In the case when  $\gamma$  is not identically equal to zero on the whole boundary, these conditions turn out to be insufficient for the unique determination of the solution of the system (1)–(2). Indeed, where  $\gamma < 0$ , “fluid” flows into the interior of the domain, which therefore is not a closed part of the atmosphere and whose motion is influenced by motions outside it.

Let  $S^-$  be that part of the boundary  $S$  where  $\gamma < 0$ . We shall assume that

$$\omega(\mathbf{x}, t)|_{S^-} = \pi(\mathbf{x}, t), \quad (5)$$

where  $\omega(\mathbf{x}, t) = \partial v/\partial x - \partial u/\partial y$  is the vertical component of the velocity vorticity. Condition (5) for a special case was proposed by N. E. Kochin (4), and later V. I. Yudovich (5) proved an existence and uniqueness theorem for the solution of problem (1)–(5), but under the condition that the right-hand side  $\mathbf{F}$  does not depend on the unknown functions. In the next section it will be shown that, in the case when  $\mathbf{F} = \mathbf{V} \times \mathbf{l}$ , it is possible to carry out the proof of V. I. Yudovich’s theorem with minor changes. For this purpose we introduce a new function  $\omega_a = \omega + l$ , the absolute vorticity. The equation for  $\omega_a$  is obtained by applying the curl operation to both sides of equation (1) and taking (2) into account:

$$\partial\omega_a/\partial t + \mathbf{V} \cdot \nabla\omega_a = 0. \quad (6)$$

We replace condition (5) by

$$\omega_a|_{S^-} = \pi + l \equiv \pi_a. \quad (7)$$

It is well known that the velocity vector is uniquely reconstructed from the vorticity and divergence specified in a domain, and also from the normal component of the velocity vector on the boundary (6). In our case we have the system for determining the velocity vector in the domain  $\Omega$

$$\partial v/\partial x - \partial u/\partial y = \omega_a(\mathbf{x}, t) + l(\mathbf{x}), \quad \partial u/\partial x + \partial v/\partial y = 0 \quad (8)$$

and the additional condition (3) on the boundary  $S$ . This system is easily reduced to two second-order equations for the functions  $\varphi$  and  $\psi$ , which are connected with the velocity vector in the following way:

$$u = -\partial\psi/\partial y + \partial\varphi/\partial x, \quad v = \partial\psi/\partial x + \partial\varphi/\partial y. \quad (9)$$

Substituting (9) into (8) and considering that on the boundary of the domain the function  $\psi$  is equal to zero, we obtain

$$\Delta\varphi = 0 \text{ inside } \Omega, \quad \partial\varphi/\partial n|_S = \gamma; \quad (10)$$

$$\Delta\psi = \omega_a - l \text{ inside } \Omega, \quad \psi|_S = 0, \quad (11)$$

where  $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ .

p.3. In order that the solution of problem (1)–(5) satisfy the equations in the classical sense, it is necessary to impose additional conditions on the boundary of the domain and to require compatibility of the initial and boundary conditions (3)–(5). We shall assume the following to be fulfilled:

A. The boundary of the domain is a contour of class  $C^{(4)}$ .

B. One has

$$\mathbf{a} \cdot \mathbf{n}|_S = \gamma(\mathbf{x}, 0), \quad \oint_S \gamma ds = 0, \quad \omega_a(\mathbf{x}, 0)|_{S^-} = \pi_a(\mathbf{x}, 0).$$

C. The quantities  $\frac{1}{\gamma} \frac{\partial \pi_a}{\partial t}$ ,  $\frac{1}{\gamma} \frac{\partial \pi_a}{\partial s}$  are bounded on the contour  $S^-$ .

D. At the moment  $t = 0$  on the contour  $S^-$  we have

$$\partial \pi_a / \partial t + \gamma \partial \omega_a / \partial n + a_s \partial \omega_a / \partial s = 0,$$

where  $a_s = \mathbf{a} \cdot \mathbf{s}$ ;  $\mathbf{s}$  is the vector tangent to the contour.

Consider the following iterative process. Regarding the velocity vector  $\mathbf{V}^n(\mathbf{x}, t)$  as known in the domain and on the interval  $[0, T]$ , we integrate (6) under condition (7), obtaining  $\omega_a^{n+1}(\mathbf{x}, t)$ . As was shown by V. I. Yudovich<sup>(5)</sup>, under conditions A–D the solution of (6)–(7) exists and is unique. Then from (10), (11), and (9) we determine  $\mathbf{V}^{n+1}(\mathbf{x}, t)$ , and so on.  $\omega_a^n$  converges strongly to a certain function  $\omega_a$  uniformly on the interval  $[0, T]$ . For the proof we apply the method proposed by V. I. Yudovich<sup>(5)</sup>, taking into account that the basic a priori estimates for the absolute vorticity and the velocity vector have the form

$$\|\omega_a(\mathbf{x}, t)\|_{L_p(\Omega)} \leq \left[ \|\omega_a(\mathbf{x}, 0)\|_{L_p(\Omega)}^p + \int_0^t \int_{S^-} |\gamma| |\pi_a|^p ds d\tau \right]^{1/p},$$

$$\|\mathbf{V}(\mathbf{x}, t)\|_{W_p^{(r)}(\Omega)} \leq cp \left( \|\omega_a(\mathbf{x}, t)\|_{W_p^{(r-1)}(\Omega)} + \|\gamma\|_{B^{l,\lambda}(S)} + \|l(\mathbf{x})\|_{W_p^{(r-1)}(\Omega)} \right).$$

The uniqueness of the solution and the well-posedness of problem (6)–(7) follow from the inequality

$$\|\omega_a^1(\mathbf{x}, t) - \omega_a^2(\mathbf{x}, t)\|_{L_p(\Omega)} \leq \|\omega_a^1(\mathbf{x}, 0) - \omega_a^2(\mathbf{x}, 0)\|_{L_p(\Omega)} \exp(ct),$$

where  $\omega_a^1$  and  $\omega_a^2$  are two solutions of (6) and (4), corresponding to different initial conditions.

From the existence and uniqueness (under conditions A–D) of the function  $\omega_a$  satisfying system (6)–(7), there follows the existence and uniqueness of the solution of problem (1)–(5).

p.4. Let us dwell on the question of computing the geopotential. After simple transformations from (1) it is easy to obtain

$$\partial \mathbf{V} / \partial t + \mathbf{V} \times \vec{\omega}_a = -\nabla E, \quad (12)$$

where  $\vec{\omega}_a = \{0, 0, \omega_a\}$ ,  $E = \Phi + V^2/2$ . Taking (2) into account, from (12) we obtain the equation

$$\Delta E = -\nabla \cdot (\mathbf{V} \times \vec{\omega}_a), \quad (13)$$

which is the equation for determining  $\Phi$ . We also obtain the boundary condition from (12), by multiplying scalarly by the vector  $\mathbf{S} = \{\cos(\widehat{ny}), -\cos(\widehat{nx}), 0\}$ , which is tangent to the boundary  $S$ . We have

$$\partial E / \partial s = -\frac{\partial \mathbf{s} \cdot \mathbf{V}}{\partial t} - \mathbf{s} \cdot (\mathbf{V} \times \vec{\omega}_a). \quad (14)$$

Integrating (14) along the contour  $S$ , we obtain

$$E(s) = E_0 - \int_0^s \left\{ \frac{\partial \mathbf{s} \cdot \mathbf{V}}{\partial t} + \gamma \omega_a \right\} ds; \quad (15)$$

$E_0$  is a constant, to within which the value of the geopotential is determined from system (1)–(2).

Received  
9 VI 1967

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*Note: Figure translations are in progress. See original paper for figures.*

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