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PERIODICALLY
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Abstract

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ON THE THEORY OF STEADY WAVES OF FINITE AMPLITUDE CAUSED BY PRESSURE PERIODICALLY DISTRIBUTED OVER THE SURFACE OF A FLOW OF A HEAVY FLUID OF INFINITE DEPTH

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The problem considered here was first posed and approximately solved by L. N. Sretenskii ⁽¹⁾. Specifying the pressure on the free surface in the form of a single first harmonic, he calculated, without investigating convergence, three terms of series in powers of a small parameter giving the solution of the problem. In solving it he used an effective method developed by him, consisting in the joint use of Eulerian and Lagrangian variables.

We have considered the exact solution of the problem, specifying the pressure on the surface by a certain infinite trigonometric series. We also investigated the special case in which the wavelength of the specified pressure coincides with the length of the steady free wave corresponding to the given flow velocity and to a constant pressure on the surface. Here we briefly present the results obtained by us.

Consider a plane-parallel steady motion of an ideal incompressible heavy fluid bounded only above by a free surface, on which the pressure $p = p_0(x)$ is a given periodic function of the horizontal coordinate x . Suppose that the flow moves from left to right with constant velocity c at infinite depth. Since on the surface the pressure is a periodic function of x , the surface takes the form of a stationary periodic wave in coordinates associated with a progressive wave having velocity $-c$.

Let the sought wave and the pressure $p_0(x)$ have the same symmetry with respect to the vertical through the crest. Let the axis Oy coincide with the axis of symmetry and direct it vertically upward. Take as the origin of coordinates O the point of intersection of the axis Oy with the free surface, and direct the axis Ox to the right. We take the plane of flow xOy as the plane of the complex variable $z = x + iy$. Introduce the usual notation: φ is the velocity potential; ψ is the stream function; $w = \varphi + i\psi$ is the complex velocity potential.

To derive from the boundary condition the basic equation of the problem, we first map conformally the region occupied by one wave and constituting an infinite vertical half-strip bounded above by a wave-shaped curve onto the half-strip $|\varphi| \leq \frac{1}{2}c\lambda$, $0 \leq \psi \leq \infty$, and then this half-strip onto the interior of the unit circle with center at zero in the plane $u = u_1 + iu_2$. It is assumed here that the wavelength λ coincides with the period of the function $p_0(x)$. As is known, the latter mapping is given by the formula

$$w = \frac{c\lambda}{2\pi i} \ln u, \quad (1)$$

where the wave profile is mapped onto the circumference of the unit circle with a cut along the radius $\arg u = \pi$.

The mapping of the circle $|u| \leq 1$ onto the region of one wave in the z -plane is determined from the relation

$$\frac{dz}{du} = -\frac{\lambda}{2\pi i} \frac{f(u)}{u}, \quad (2)$$

where

$$f(u) = 1 + \sum_{k=1}^{\infty} a_k u^k.$$

The coefficients a_k are real, since the wave is symmetric with respect to the Oy axis and $a_0 = 1$, for the velocity of the flow at infinity is directed along the Ox axis and is equal to c .

As usual, introducing the function

$$\omega(u) = \Phi + i\tau = -i \ln f(u), \quad (3)$$

we find from (3) and (2) that, for $u = e^{i\theta}$ (θ is the angle of the radius vector with the u_1 axis),

$$\frac{dx}{d\theta} + i \frac{dy}{d\theta} = -\frac{\lambda}{2\pi} e^{-\tau(\theta)} (\cos \Phi + i \sin \Phi). \quad (4)$$

From formulas (3), (2), and (1) it follows that everywhere in the flow the function Φ is equal to the angle of the velocity vector \mathbf{q} with the Ox axis, and that

$$q = |\mathbf{q}| = c \exp(\tau). \quad (5)$$

Taking Bernoulli's integral for the surface, where $p = p_0(x)$, we differentiate it with respect to θ ; then, using (4) and (5), we obtain a differential relation, whose integration introduces the constant of integration

$$\mu = \frac{3g\lambda}{2\pi c^2} \exp[-3\tau(0)]; \quad (6)$$

here g is the acceleration due to gravity. Taking the logarithmic derivative of both sides of the indicated integral relation, we obtain

$$\frac{d\tau}{d\theta} = \frac{1}{3}\mu \left(\sin \Phi + \frac{1}{\rho g} \frac{dp_0}{dx} \cos \Phi \right) \left/ \left[1 + \mu \int_0^\theta \left(\sin \Phi + \frac{1}{\rho g} \frac{dp_0}{dx} \cos \Phi \right) d\eta \right] \right.; \quad (7)$$

here ρ is the density of the liquid.

This equality gives the connection between the functions $\tau(\theta)$ and $\Phi(\theta)$ on the circumference $|u| = 1$.

Since the function $\tau(\theta)$ is symmetric with respect to the real axis, $\tau(\theta) = \tau(2\pi - \theta)$. Hence the validity of the well-known Dini relation follows:

$$\Phi(\theta) = -\frac{1}{2\pi} \int_0^{2\pi} \frac{d\tau}{d\eta} \ln \left| \frac{\sin(\eta - \theta)/2}{\sin(\eta + \theta)/2} \right| d\eta. \quad (8)$$

From (7) and (8) we finally have

$$\Phi(\theta) = -\frac{\mu}{6\pi} \int_0^{2\pi} \frac{\sin \Phi + \frac{1}{\rho g} \frac{dp_0}{dx} \cos \Phi}{1 + \mu \int_0^\eta \left(\sin \Phi + \frac{1}{\rho g} \frac{dp_0}{dx} \cos \Phi \right) d\eta_1} \ln \left| \frac{\sin \frac{\eta - \theta}{2}}{\sin \frac{\eta + \theta}{2}} \right| d\eta. \quad (9)$$

This is the integral equation of the problem.

From this equation, for $p_0 = \text{const}$, one obtains the well-known equation of A. I. Nekrasov^(2,3). In contrast to A. I. Nekrasov's equation, equation (9) is not homogeneous, since the function $\Phi \equiv 0$ does not satisfy it.

In solving equation (9) we assume that

$$\frac{1}{\rho g} \frac{dp_0}{dx} = \sum_{n=1}^{\infty} \varepsilon^n d_n \sin n\theta, \quad (10)$$

where ε is a small positive dimensionless parameter, and d_n are prescribed real numbers, with the series $\sum_{n=1}^{\infty} \varepsilon^n d_n$ converging in a circle of radius $\varepsilon_0 > 0$. We note that in the original problem p_0 is a prescribed periodic function of x .

It can be shown, however, that the solution of the problem under consideration, under condition (10), corresponds to prescribing the series

$$\frac{1}{\rho g} \frac{dp_0}{dx} = - \sum_{n=1}^{\infty} \varepsilon^n \tilde{c}_n \sin \frac{2\pi}{\lambda} x, \quad \text{where } \tilde{c}_n = \sum_{m=0}^{\infty} \varepsilon^m \tilde{c}_{mn}.$$

In this case either the coefficients \tilde{c}_{0n} may be regarded as prescribed and the d_n determined from them, or, conversely, the coefficients \tilde{c}_{mn} ($m = 1, 2, \dots$) are determined in terms of d_n .

Let us transform equality (6). On the circle $|u| = 1$, the function $\tau(\theta)$ is even, while the function $\Phi(\theta)$ is odd. Therefore

$$\Phi(\theta) = \sum_{k=1}^{\infty} b_k \sin k\theta, \quad \tau(\theta) = - \sum_{k=1}^{\infty} b_k \cos k\theta; \quad (11)$$

$b_0 = 0$, since $\omega(0) = 0$ (see (2) and (3)). Hence (6) takes the form

$$\mu = \mu_0 \exp \left[3 \sum_{k=1}^{\infty} b_k \right], \quad \text{where } \mu_0 = \frac{3g\lambda}{2\pi c^2}; \quad (12)$$

here, by the condition of the problem, λ and c are regarded as fixed.

Equation (12) serves to determine the parameter μ . We note that prescribing μ determines the velocity at the crest of the wave (see (5) and (6)).

Putting

$$\Psi(\theta) = \left[1 + \mu \int_0^\theta (\sin \Phi + Q(\eta) \cos \Phi) d\eta \right]^{-1}, \quad \text{where } Q(\eta) = \frac{1}{\rho g} \frac{dp_0}{dx}, \quad (13)$$

we reduce, as in the case of Nekrasov's equation (2, 3), equation (9) to an equivalent system of two equations with unknown functions $\Phi(\theta)$ and $\Psi(\theta)$:

$$\Phi(\theta) = \mu \int_0^{2\pi} K(\eta, \theta) \Psi(\theta) [\sin \Phi + Q(\eta) \cos \Phi] d\eta, \quad (14)$$

$$\Psi(\theta) = 1 - \mu \int_0^\theta \Psi^2(\eta) [\sin \Phi + Q(\eta) \cos \Phi] d\eta, \quad (15)$$

where

$$K(\eta, \theta) = -\frac{1}{6\pi} \ln \left| \frac{\sin(\eta - \theta)/2}{\sin(\eta + \theta)/2} \right|.$$

From the expression for the kernel it follows that

$$K(\eta, \theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\eta \sin n\theta}{3n}. \quad (16)$$

Thus, the normalized eigenfunctions $\varphi_n(\theta)$ and eigenvalues ν_n of the kernel (16) have the form

$$\varphi_n(\theta) = \frac{\sin n\theta}{\sqrt{\pi}}, \quad \nu_n = 3n. \quad (17)$$

The problem has been reduced to determining the functions $\Phi(\theta)$ and $\Psi(\theta)$ from the system of equations (14), (15) and the parameter μ from equation (12). In solving it one has to consider two cases: in the first case $\mu_0 \neq \nu_n$, in the second $\mu_0 = \nu_n$.

In the first case the solution $\Phi(\theta, \varepsilon)$, $\Psi(\theta, \varepsilon)$, and $\mu(\varepsilon)$ is constructed in the form of series in integral powers of the parameter ε . In the second case, as an example we considered the value $\mu_0 = \nu_1$. Here the solution is obtained in the form of series in powers of $\varepsilon^{1/3}$. In both cases we prove that these series converge absolutely and uniformly for $0 \leq \theta \leq 2\pi$ and small values $|\varepsilon| < \varepsilon_1 \leq \varepsilon_0$, and that the coefficients of these series are determined uniquely. Applying the Lyapunov-Schmidt methods with the use of the Newton diagram (4), we also prove that the series considered here give the unique solution of the problem, small relative to ε , continuous in θ .

The wave profile in parametric form $x(\theta, \varepsilon)$ and $y(\theta, \varepsilon)$ is determined from relation (4), into which one must substitute the found $\Phi(\theta, \varepsilon)$ and $\tau(\theta, \varepsilon)$, computed by the second formula (11). Eliminating θ from the parametric equation, we obtain the profile equation in the form $y = y(\theta, \varepsilon)$.

We give the approximate profile equations in both cases, accurate up to terms of the third order, putting $k = 2\pi/\lambda$.

In the case $\mu_0 \neq \nu_n$:

$$y(x, \varepsilon) = \frac{1}{k} \left\{ \varepsilon C_{11} (\cos kx - 1) + \frac{\varepsilon^2}{2} \left(\frac{1}{2} C_{11}^2 - C_{22} \right) (1 - \cos 2kx) + \right. \\ \left. + \frac{\varepsilon^3}{6} \left[6C_{13} (\cos kx - 1) + \frac{9}{2} C_{11} C_{22} (\cos kx - \cos 3kx) + \right. \right.$$

$$+ \left(C_{33} + C_{11}C_{22} + \frac{1}{6}C_{11}^3 \right) (\cos kx - 1) \Big\},$$

where

$$C_{11} = \frac{\mu_0 d_1}{3 - \mu_0}, \quad C_{22} = \left[d_2 + \frac{9\mu_0 d_1^2}{2(3 - \mu_0)^2} \right] \frac{\mu_0}{6 - \mu_0}, \quad C_{13} = \frac{\mu_0}{3 - \mu_0} C'_{13}, \quad C_{33} = \\ = \frac{\mu_0}{9 - \mu_0} C'_{33};$$

here C'_{13} and C'_{33} are linear functions of C_{11}^3 , C_{11}^2 , C_{22} , d_1 , and d_2 .

In the case $\mu_0 = \nu_1$:

$$y(x, \varepsilon) = \frac{1}{k} \left\{ -\varepsilon^{1/3} d_1^{1/3} (\cos kx - 1) - \frac{1}{2} \varepsilon^{2/3} d_1^{2/3} (1 - \cos 2kx) + \right. \\ \left. + \frac{\varepsilon}{6} d_1 \left[10^{31}/48 (\cos kx - 1) - \frac{27}{4} (\cos kx - \cos 3kx) - 9 (\cos 3kx - 1) \right] \right\}.$$

By the condition of the problem the origin of coordinates is placed at the crest of the wave. Therefore, from an analysis of the leading terms in the formulas for $y(x, \varepsilon)$ and assuming $3 < \mu_0 < 6$, we conclude that one must take $d_1 < 0$. We also note that $\mu_0 = \nu_1$ is the special case which was mentioned at the beginning of the article.

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Note: Figure translations are in progress. See original paper for figures.

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