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Abstract

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MATHEMATICS

Academician of the Academy of Sciences of the BSSR F. D. GAKHOV

ON A NONLINEAR BOUNDARY-VALUE PROBLEM GENERALIZING THE RIEMANN BOUNDARY-VALUE PROBLEM

A. Suciú (¹) posed a very general nonlinear homogeneous boundary-value problem. In a $(p + 1)$ -connected domain D^+ , bounded by a contour L consisting of smooth closed curves L_0, L_1, \dots, L_p , and its complement to the full plane D^- , determine a piecewise-analytic function $\Phi(z) = \{\Phi^+(z), \Phi^-(z)\}$ satisfying on L the boundary condition

$$[\Phi^+(t)]^{\alpha(t)} = G(t)[\Phi^-(t)]^{\beta(t)}, \quad (1)$$

where $G(t)$ is a given function satisfying the Hölder condition and not vanishing, while the exponents $\alpha(t), \beta(t)$ do not vanish and are the boundary values of functions analytic respectively in D^+, D^- . A particular case of this problem is the well-known linear Riemann boundary-value problem (²), as well as the nonlinear cases considered by G. V. Arzhanov (³) (one of the exponents is equal to a positive integer and the other to unity), and also by G. P. Cherepanov (⁴) ($\alpha = -\beta = 1$).

In the cited work of A. Suciú, certain steps were taken toward the investigation of the problem. However, the excessive generality in the formulation of the problem did not allow the author to obtain decisive results. Only the case in which the exponents are equal to reciprocals of integers may be regarded as investigated. Here branch points of the solution cannot arise, and therefore the difficulties of the investigation disappear. For other exponents the author treats $\ln G(t)$ as a continuous single-valued function, which is possible only when $\text{Ind } G(t) = 0$. In this simplest case branch points are also absent.

In the present note, using a necessary solvability condition following from the argument principle for analytic functions, we indicate a class of exponents α, β in which a solution of the problem is possible. For one admissible case we shall give an exhaustive investigation.

Consider the boundary-value problem (1), in which we put

$$\text{Ind } G(t) = \frac{1}{2\pi} \{\arg G(t)\}_L = \varkappa.$$

Together with A. Suci, we shall regard $\alpha(t), \beta(t)$ as boundary values of functions analytic respectively in D^\pm , and for the time being retain the remaining assumptions of the author. Denote by N_+ , respectively, the numbers of zeros of the sought functions $\Phi^\pm(z)$ in their domains of definition, and suppose temporarily that there are no zeros on the contour. Taking indices of both sides of equality (1), we shall have

$$N_+ \alpha(t) + N_- \beta(t) = \varkappa. \quad (2)$$

The last equality is the boundary condition of the Riemann problem for finding functions $\alpha(z), \beta(z)$ analytic respectively in the domains D^\pm . The coefficient and the free term of the problem will be constants; its general solution is expressed by the formulas

$$\alpha(z) = C/N^+, \quad \beta(z) = (\varkappa - C)/N^-. \quad (3)$$

We obtain the following conclusion:

Theorem 1. *In the boundary-value problem (1), the exponents $\alpha(t), \beta(t)$, under the condition of their analytic continuation from the contour respectively into the domains D^\pm , can only be constants determined by formulas (4), where C is an arbitrary constant.*

For given α, β , formulas (3) or relation (2) determine the admissible number of zeros of the solution. From the analysis of any of these one can derive one more conclusion.

Theorem 2. *If the constant exponents α, β have the same sign, then the total number of zeros $N_+ + N_-$ of the solution $\Phi^\pm(z)$ in the domains D^\pm is finite; when α, β have different signs, it may grow without bound.*

Corollary. *If $\varkappa < 0$, then in the class of analytic functions the problem has only the zero solution.*

We shall consider here the case when α, β are positive integers. Here, as will be shown below, the problem admits an exhaustive solution.

Thus, let in the boundary condition (1) α, β be positive integers. Assume, for simplicity, that the contour L is a simple smooth closed curve dividing the plane into two simply connected domains D^+, D^- , the first of which contains the origin, and the second the infinitely distant point. We use the representation arising from the solution of the linear problem

$$t^{-\chi}G(t) = \frac{e^{\Gamma^+(z)}}{e^{\Gamma^-(z)}}, \quad \Gamma(z) = \frac{1}{2\pi i} \int_L \frac{\ln[\tau^{-\chi}G(\tau)]}{\tau - z} d\tau. \quad (4)$$

Using, analogously to how this is done in solving the linear problem, the theorem on analytic continuation in adjacent domains and the generalized Liouville theorem, we obtain that the solution, if it exists, necessarily has the form

$$\Phi^+(z) = (e)^{\frac{1}{\alpha}\Gamma^+(z)}[P(z)]^{\frac{1}{\alpha}}, \quad \Phi^-(z) = e^{\frac{1}{\beta}\Gamma^-(z)}[z^{-\chi}P(z)]^{\frac{1}{\beta}};$$

where $P(z)$ is some polynomial of degree χ .

In contrast to the linear problem, the polynomial $P(z)$ cannot be chosen arbitrarily. In order that the solution be analytic (that branch points be absent), it must be chosen so that in the domains D^\pm its zeros have multiplicities respectively α, β . Hence it follows that $P(z)$ must have the form

$$P(z) = [P_{N_+}(z)]^\alpha [P_{N_-}(z)]^\beta, \quad (5)$$

where P_{N_+}, P_{N_-} are arbitrary polynomials of degrees N_+, N_- with zeros respectively in the domains D^+, D^- ; for one of them (for example, for P_{N_-}) the coefficient of the highest degree may be set equal to unity.

Consequently, the solution of the problem, if it exists, necessarily has the form

$$\Phi^+(z) = e^{\frac{1}{\alpha}\Gamma^+(z)} P_{N_+}(z) [P_{N_-}(z)]^{\frac{\beta}{\alpha}},$$

$$\Phi^-(z) = e^{\frac{1}{\beta}\Gamma^-(z)} [z^{-N_+} P_{N_+}(z)]^{\frac{\alpha}{\beta}} z^{-N_-} P_{N_-}(z), \quad (6)$$

where $\Gamma(z)$ is determined by formula (4). Hence follows

Theorem 3. *The solution of problem (1), if it exists, depends (nonlinearly) on $N_+ + N_- + 1$ arbitrary parameters.*

The number of admissible zeros N_+, N_- is determined by equality (3). The latter is an indeterminate equation in integers with respect to N_+, N_- . For solvability of the boundary-value problem (2) under consideration, it is obviously necessary that equation (3) be solvable in nonnegative integers. From the theory of indeterminate equations it follows

Theorem 4. *For the solvability of the boundary value problem (1) it is necessary that the greatest common divisor of the exponents $\alpha, \beta, d = D(\alpha, \beta)$, be a divisor of the index \varkappa .*

The latter condition, as is known, is also sufficient for the solvability of equation (2) in integers. However, among these solutions there may be none in which

both variables are nonnegative. Therefore the conditions of Theorem 4 are not sufficient for the existence of analytic solutions of problem (1). If one changes the original formulation of the problem and admits also solutions with polar singularities, then equation (2) must be replaced by the equation

$$\alpha(N_+ - P_+) + \beta(N_- - P_-) = \varkappa, \quad (7)$$

where P_{\pm} is the number of poles, respectively, in the domains D^{\pm} . In such a broader formulation the following result holds.

Theorem 5. *If the condition of Theorem 4 is satisfied, there exists an infinite set of solutions of the boundary value problem (1) having a certain number of poles. Among these solutions one can choose an infinite set such that one of the unknown functions (Φ^+ or Φ^-) is analytic.*

Let us now return to the original formulation—the search for analytic solutions of problem (1). It is not difficult to see that the following is true.

Theorem 6. *For the solvability of the boundary value problem (1) in analytic functions it is necessary and sufficient that the indeterminate equation (2) be solvable in nonnegative integers.*

We indicate some criteria for the existence of nonnegative integer solutions of equation (2), or, what is the same, for the existence of analytic solutions of the boundary value problem (1). We shall assume that the necessary solvability conditions ($\varkappa \geq 0, D(\alpha, \beta, \varkappa) = 1$) are satisfied.

Suppose that, by the Euclidean algorithm or by one of the methods for solving congruences, we have found some particular integer solution (x_1, y_1) of equation (2). Assume that $-y_1/\alpha \leq x_1/\beta$. The general solution in integers of equation (2) can be represented in parametric form as follows:

$$N_+ = x_1 - \beta t, \quad N_- = y_1 + \alpha t,$$

where t may take any integer values. From the condition $N_+, N_- \geq 0$ it follows that

$$-[y_1/\alpha] \leq t \leq [x_1/\beta].$$

The admissible values are the integral values of t satisfying the last inequality. To each of them there will correspond an integral nonnegative solution of equation (2); by formulas (6), the latter will correspond to analytic solutions of the original problem (1). Thus the problem will have $[x_1/\beta] - [y_1/\alpha] + 1$ solutions. Each of them contains $N_+ + N_- + 1$ arbitrary parameters, where for (N_+, N_-) one must successively take all integral nonnegative solutions of the indeterminate equation (2).

If \varkappa/α or \varkappa/β is not very large, then the method of trial will be quite an effective means of finding analytic solutions of problem (1). Suppose, for example, that $\alpha > \beta$; if to some integer value N_+ lying in the interval $0 \leq N_+ \leq \varkappa/\alpha$ there corresponds an integral nonnegative value of the other unknown N_- , then such a pair (N_+, N_-) will generate an analytic solution of problem (1). In this way we obtain all possible solutions. If no integral nonnegative values of N_- exist, then problem (1) in analytic functions is unsolvable (the trivial zero solution is not counted). From this it is easy to obtain one simple criterion of unsolvability. For an index from the interval

$$0 < \varkappa < \min(\alpha, \beta)$$

problem (1) has no analytic solutions.

Example. Let $\alpha = 5$, $\beta = 11$, $\varkappa = 198$. The admissible numbers of zeros N_+, N_- are determined by the equation $5N_+ + 11N_- = 198$. We replace the latter by the congruence $11N_- \equiv 198 \pmod{5}$. Each time discarding in the right-hand side ...

terms congruent modulo, using Fermat's theorem, we obtain $N_- \equiv 3 \cdot 11^3 \equiv 3$, and the corresponding value is $N_+ = 33$. The pair $(33, 3)$ is a particular solution of the equation. The general solution is $N_+ = 33 - 11t$, $N_- = 3 + 5t$. The nonnegativity condition for N_+, N_- gives the inequality $-3/5 < t \leq 3$. Consequently, the admissible values are $t = 0, 1, 2, 3$. They correspond to 4 solutions of the indeterminate equation in nonnegative integers

$$(33, 3); \quad (22, 8); \quad (11, 13); \quad (0, 18).$$

Hence, for the boundary-value problem $[\Phi^+(t)]^5 = G(t)[\Phi^-(t)]^{11}$ with $\text{Ind } G(t) = 198$, by formulas (6) we obtain 4 distinct solutions, of which, to save space, we write down only the first:

$$1. \begin{cases} \Phi^+(z) = e^{1/5 \Gamma^+(z)} P_{33+}(z) [P_{3-}(z)]^{11/5}, \\ \Phi^-(z) = e^{1/11 \Gamma^-(z)} [z^{-33} P_{33+}(z)]^{5/11} z^{-3} P_{3-}. \end{cases}$$

The solutions contain, respectively, 37, 31, 25, and 19 arbitrary parameters. The method of trials for obtaining the solutions of the indeterminate equation would be less advantageous here, since it would be necessary to check $198/11 + 1 = 19$ numbers.

Let us now remove the restriction imposed at the beginning of the investigation that the solutions do not vanish on the contour. For this it will be necessary to allow the zeros of the polynomials P_{N_+}, P_{N_-} to lie not only in the domains D^\pm , but also on the contour. From the condition $G \neq 0$ and the boundary condition it follows that a zero on the contour is common to both functions $\Phi^\pm(z)$; its

admissible order is determined by formulas (6). All the arguments, including relation (2), remain valid if, starting from a solution that does not vanish, as from an auxiliary one, we subsequently assign an admissible zero on the contour in half-order to each of the functions $\Phi^\pm(z)$.

Remark 1. The functional properties of the coefficient $G(t)$ were used by us only once, namely in formulas (4). It follows that for $G(t)$ the admissible class of functions is that in which the solution of the linear Riemann boundary-value problem with coefficient $G(t)$ is possible. The most general of such classes is apparently the class, introduced by I. B. Simonenko [5], of bounded measurable functions with the additional condition that, at points of discontinuity, the oscillation of the argument is less than $2\pi - \delta$ ($\delta > 0$).

Remark 2. The case when the exponents α, β are rational numbers presents nothing new. By raising both sides of the boundary condition (1) to the power equal to the common denominator of the exponents α, β , it is immediately reduced to the case considered of integer exponents. On the contrary, the case of a multiply connected domain, in contrast to what occurs in linear problems, may in the general case present fundamental difficulties.

Belorussian State University
named after V. I. Lenin

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