ON THE SUMMATION OF SERIES IN EIGENFUNCTIONS

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Abstract

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MATHEMATICS

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ON THE SUMMATION OF SERIES IN EIGEN-FUNCTIONS

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Consider, in N-dimensional space, a normal domain g with boundary Γ and a complete orthonormal system $\{u_n(x)\}$ of eigenfunctions of the Laplace operator for the first boundary-value problem

$$\Delta u + \lambda u = 0, \quad x \in g, \qquad u\big|_\Gamma = 0.$$

We define a continuous triangular summation method generated by a function $\varphi_\lambda(t) = 0$ for $t \geq \lambda$, absolutely continuous for $t \geq 0$, and satisfying the condition

$$\lim_{\lambda \to \infty} \varphi_\lambda(t) = 1.$$

A series $\sum b_n$ will be called summable by the method φ to the number s if there exists the limit

$$\lim_{\lambda \to \infty} \sum_{\lambda_n < \lambda} \varphi_\lambda(\lambda_n) b_n = s.$$

The present note is devoted to the study of the properties of the kernel of this summation method

$$\Phi(x, y, \lambda) = \sum_{\lambda_n < \lambda} \varphi_\lambda(\lambda_n) u_n(x) u_n(y).$$

For the Riesz method (R, λ_n, α) exact results are known ([1–4]), which make it possible to sum the Fourier series of $f(x) \in L_2(g)$. In the case of multiple Fourier series, E. Stein ([5]) obtained results for $f(x) \in L_p(g)$, $p \geq 1$.

For any $\xi > 0$ and integer $k \geq 0$, define

$$d_k(\xi) = \left(\frac{\lambda}{2\pi}\right)^{N/2} \frac{2^{-k}}{k!} \int_0^1 \varphi_\lambda(\lambda t^2) \frac{J_{(N-2)/2+k}(\xi t)}{\xi^{(N-2)/2+k}} t^{N/2+k}\, dt.$$

For an arbitrary fixed $R > 0$, denote by g_R the set of points of the domain g whose distance from Γ is greater than R, and by $D_s(r, \lambda)$ the function which, for $r > R$, is equal to zero and, for $r \leq R$, is defined by the equality

$$D_s(r, \lambda) = \sum_{k=s}^{\infty} d_k(R\sqrt{\lambda})\lambda^k (R^2 - r^2)^k, \qquad s = 0, 1, 2, \ldots$$

Then from the mean-value formula (([1], p. 230) there follows the equality

$$\Phi(x, y, \lambda) = D_s(r_{xy}, \lambda) + \sum \gamma_n u_n(x) u_n(y), \tag{1}$$

where

$$\gamma_n = (2\lambda)^s s!\,(2\pi)^{N/2} \int_R^{\infty} d_s(r\sqrt{\lambda})\, \frac{J_\nu(r\sqrt{\lambda_n})}{\lambda_n^{\nu/2}}\, r^{\nu+1}\, dr, \qquad \nu = (N-2)/2 + s.$$

Note that $D_0(r_{xy}, \lambda)$ for $r_{xy} \leq R$ coincides with $\Phi^*(x, y, \lambda)$, the kernel of summation by the method φ of the expansion into an ordinary N-fold Fourier integral.

Theorem 1. Let the summability function $\varphi_\lambda(t)$ have a derivative of order $l > N/p,\ 1 \leq p \leq 2$, satisfying the condition

$$\left|\varphi_\lambda^{(l)}(t)\right| < c\lambda^{-l}. \tag{2}$$

Then, uniformly with respect to $x \in g_R$, the formula

$$\|\Phi(x, y, \lambda) - \Phi^*(x, y, \lambda)\|_{L_q(g)} = o(1)$$

holds. The norm is taken with respect to $y \in g,\ q = p/(p-1)$.

To prove the theorem it is enough to estimate the right-hand side of (1), which is not difficult to do by using the properties of fractional-order kernels ([6]). The indicated scheme corresponds to the method of Minakshisundaram ([1,2]).

Theorem 1 asserts the equisummability of expansions in series in eigenfunctions and in the ordinary Fourier integral. Following S. Bochner ([7]), one can establish the summability of the expansion in the Fourier integral of an arbitrary function $f(x) \in L(g)$ to the value of the function at every Lebesgue point in the case where φ satisfies condition (2) with $l > (N+1)/2$.

Corollary. Let p belong to the interval $1 \leq p \leq 2N/(N+1)$, and let (2) hold for $\varphi_\lambda(t)$ with $l > N/p$.

Then the Fourier series of an arbitrary function $f(x) \in L_p$ is summed by the method φ to the value of the function at every Lebesgue point.

For the Riesz method, from equality (1) one can obtain a more precise result.

Theorem 2. If $f(x) \in L_p(g),\ 1 \leq p \leq 2$, then its Fourier series in eigenfunctions is summed to it by the Riesz method (R, λ_n, α) of order $\alpha > N/p - \frac{1}{2}$ at every Lebesgue point.

If $f(x) \in L_p(g),\ 1 < p < 2$, then the same assertion remains valid for $\alpha = N/p - \frac{1}{2}$.

The author considers it necessary to note that analogous results can be obtained from the Tauberian theorem of Levitan–Marchenko ([8]).

In conclusion I take the opportunity to express my gratitude to Prof. V. A. Il' in for his attention to my work.

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Note: Figure translations are in progress. See original paper for figures.