ASYMPTOTIC OPTIMALITY OF CUBATURE FORMULAS WITH A BOUNDARY LAYER REGULAR IN THE SENSE OF S. L. SOBOLEV

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Abstract

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ASYMPTOTIC OPTIMALITY OF CUBATURE FORMULAS WITH A BOUNDARY LAYER REGULAR IN THE SENSE OF S. L. SOBOLEV

(Presented by Academician S. L. Sobolev on 15 V 1967)

1°. In note ([1]) it was shown that the problem of constructing a cubature formula optimal with respect to coefficients in the spaces $H_2^{(\mu)}(\Omega)$, embedded in the space C, is a linear problem. But, since in computing integrals of high multiplicity the order of the linear system from which the optimal coefficients are found will be very large, finding them directly is quite difficult. In note ([2]) we introduced into consideration much simpler formulas with a boundary layer regular in the sense of S. L. Sobolev and, under the assumption that the weight functions belong to the weight class $B_n^{(m)}$, i.e. are homogeneous functions of dimension $m > n/2$ and satisfy the system of inequalities

$$\left|D^\alpha F[\mu^{-2}(\xi)]\right| \le K|x|^{2m-n-|\alpha|}, \tag{1}$$

showed that for the square of the norm of the error functional of any such formula the following asymptotic expression holds:

$$\|l\|^2_{H_2^{(\mu)*}(\Omega)} = h^{2m} \sum_{\beta \neq 0} \frac{1}{\mu^2(\beta H^{-1})} |\Omega| + O(h^{2m+1}). \tag{2}$$

In the present note we shall show that

$$\|l\|^2_{H_2^{(\mu)*}(\Omega)} - \|l^0\|^2_{H_2^{(\mu)*}(\Omega)} = O(h^{2m+1}) \tag{3}$$

(here l denotes the functional of an arbitrary cubature formula with regular boundary layer, and l^0 denotes the functional of the formula optimal with respect to coefficients).

From relations (2) and (3) will follow the asymptotic optimality of the formulas introduced in note ([2]). We shall carry out all arguments for the weight function $\mu(\xi) = |\xi|^m$. First we shall have to clarify the properties of some discrete functions connected with the infinite matrix

$$\|v(hH(\beta-\beta'))\| \qquad \left(v(x) = F[\mu^{-2}(\xi)]\right). \tag{4}$$

2°. Recall that a discrete function is a function specified at the points of some regular lattice (we shall be interested in the lattice determined by the matrix hH, where $|H| = 1$, and h is a small parameter). Such functions, as is known, form a ring. For them one can correctly define the notions of scalar product and convolution. We shall also use the Fourier transform of discrete functions. First, from a given discrete function (we shall denote it by $\varphi_{hH}[\beta]$) we construct the lattice function

$$\varphi_{hH}(x) = \sum_{\beta} \varphi_{hH}[\beta]\delta(x - hH\beta). \tag{5}$$

and by the Fourier transform of the discrete function $\varphi_{hH}[\beta]$ we shall mean the Fourier transform of the corresponding lattice function $\varphi_{hH}(x)$, which, as is known (see (3)), will be a periodic function. All the usual relations known in the theory of Fourier transforms will then be satisfied. Now we can easily invert the infinite matrix (4). Namely, the inverse matrix

$$\|\lambda(hH(\beta-\beta'))\| \tag{6}$$

will be such that

$$v_{hH}[\beta] * \lambda_{hH}[\beta] = \delta_{hH}[\beta] = \begin{cases} 1, & \text{if } hH\beta = 0, \\ 0, & \text{if } hH\beta \neq 0. \end{cases} \tag{7}$$

Hence we find that

$$\hat{\lambda}_{hH}(\xi) = \frac{1}{\hat{v}_{hH}(\xi)} = h^n \Big/ \left(\sum_{\beta} \frac{1}{|\xi - \beta h^{-1}H^{-1}|^{2m}} \right). \tag{8}$$

Theorem 1. The function $\hat{\lambda}_{hH}(\xi)$ is a nonnegative periodic function, with fundamental period matrix $h^{-1}H^{-1}$, analytic for all real values of ξ except, possibly, for the points $\xi = \beta h^{-1}H^{-1}$, at which it has zeros (generally speaking, critical ones) of multiplicity $[2m]$.

Theorem 2. The discrete function $\lambda_{hH}[\beta]$ is representable in the form

$$\lambda_{hH}[\beta] = h^{-2m}\lambda_H[\beta], \tag{9}$$

where for $\lambda_H[\beta]$ the estimate

$$|\lambda_H[\beta]| \leq K/(1+|\beta|^2)^{m+n/2} \tag{10}$$

holds.

Theorem 3. The convolution

$$K_{hH}(x) = \lambda_{hH}(x) * v(x) \tag{11}$$

is representable in the form

$$K_{hH}(x) = K_H(x/h), \tag{12}$$

and for $K_H(x)$ the estimate

$$|K_H(x)| \leq K/(1+|x|^2)^{m+n/2} \tag{13}$$

holds.

We now introduce one special space of discrete functions $h_2^{(\hat{\lambda}_{hH})}$. We define it as the set of functions $\varphi_{hH}[\beta]$ for which the integral

$$\|\varphi_{hH}[\beta]\|^2_{h_2^{(\hat{\lambda}_{hH})}} = \int_\Omega |\hat{\varphi}_{hH}(\xi)|^2 \hat{\lambda}_{hH}(\xi)\, d\xi \tag{14}$$

is finite (here Ω denotes the fundamental torus). In this space one can naturally introduce a scalar product, turning it into a Hilbert space.

Let us also note that the norm in the space $h_2^{(\hat{\lambda}_{hH})}$, by Parseval' s equality, can be defined by the formula

$$\|\varphi_{hH}[\beta]\|^2_{h_2^{(\hat{\lambda}_{hH})}} = \sum_\beta \varphi_{hH}[\beta](\lambda_{hH}[\beta] * \varphi_{hH}[\beta]). \tag{15}$$

We shall chiefly use this discrete method of normalization. In what follows we shall have to compare two discrete functions

$$u^*_{hH}[\beta] = v_{hH}[\beta] * \rho_{hH}[\beta], \tag{16}$$

where

$$\rho_{hH}[\beta] = \begin{cases} c_\beta - c_\beta^0, & \text{if } hH\beta \in \Omega, \\ 0, & \text{if } hH\beta \notin \Omega \end{cases} \tag{17}$$

(here the c_β denote the coefficients of the formula with a regular boundary layer, and the c_β^0 the optimal coefficients) and

$$u_{hH}[\beta] = \left.(l(x) * v(x))\right|_{x=hH\beta}, \tag{18}$$

where $l(x)$ is the error functional of any formula with a regular boundary layer.

Theorem 4. Among all functions $u_{hH}[\beta] \in \hat{h}_2^{(\hat{\lambda}_{hH})}$ coinciding at the points of the domain Ω with the function $u_{hH}^*[\beta]$, this latter function gives the smallest value to the form

$$\|u_{hH}[\beta]\|^2_{\hat{h}_2^{(\hat{\lambda}_{hH})}}.$$

3°. We now outline the proof of the principal estimate (3). First of all, note that

$$\|l\|^2_{H_2^{(\mu)*}(\Omega)} - \|l^0\|^2_{H_2^{(\mu)*}(\Omega)} = \|u_{hH}^*[\beta]\|^2_{\hat{h}_2^{(\hat{\lambda}_{hH})}}. \tag{19}$$

Using Theorem 4 and Babuška' s theorem (see ([1])), we construct a majorizing form

$$\|u_{hH}^*[\beta]\|^2_{\hat{h}_2^{(\hat{\lambda}_{hH})}} \ll \|u_{hH}[\beta]\|^2_{\hat{h}_2^{(\hat{\lambda}_{hH})}}. \tag{20}$$

Relying on the properties of the error functional of a cubature formula with a regular boundary layer (see ([2])), and also on the estimates obtained above, one can show that

$$\|u_{hH}[\beta]\|^2_{\hat{h}_2^{(\hat{\lambda}_{hH})}} = O\left(h^{2m+1}\right). \tag{21}$$

The principal estimate (3) now follows from relations (19), (20), (21). We note that for integral m this estimate was obtained in his classes by S. L. Sobolev (see ([4])).

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CITED LITERATURE

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[4] S. L. Sobolev, DAN, 164, No. 2 (1965).

Note: Figure translations are in progress. See original paper for figures.