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Abstract

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UDC 517.934+62.50 **CYBERNETICS AND CONTROL THEORY**

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REGULARIZATION OF THE PROBLEM OF THE MEETING OF MOTIONS

Consider the controlled motions $y[t]$ and $z[t]$, described by the equations

$$\dot{y} = A^{(1)}y + B^{(1)}u, \tag{1}$$

$$\dot{z} = A^{(2)}z + B^{(2)}v, \tag{2}$$

where y, z are n-dimensional vectors of phase coordinates; u, v are r-vectors of control forces, whose realizations $u[t]$ and $v[t]$ are constrained by the condition

$$\|u[t]\| \leq \mu, \qquad \|v[t]\| \leq \nu \quad \text{(essentially)}; \tag{3}$$

$A^{(i)}, B^{(i)}$ are given matrices. The symbol $\|q\|$ denotes the Euclidean norm of the vector q. The vectors under consideration will be treated as column vectors; the superscript will denote transposition; the symbol $\{Q\}_{(m)}$ will denote the matrix composed of the first m rows of the matrix Q. The restriction (2) can be replaced by the more general condition

$$u[t] \in U, \qquad v[t] \in V, \tag{4}$$

where U and V are bounded, convex, and closed sets. In this case, however, concrete calculation is, generally speaking, complicated.

The problem of the meeting of motion (1) with motion (2) consists in such a choice of the control u as ensures the equality

$$\|\{y[\vartheta] - z[\vartheta]\}_{(m)}\| = 0 \tag{5}$$

at the terminal time $t = \vartheta$ (for every possible initial state $y[t_0], z[t_0]$ from the chosen domain $\mathcal{Y}$). The solution of game problems for controlled objects described by differential equations, including the solution of the meeting problem (5) under the condition that the control $u[t]$ is formed in the form

$$u[t] = u[y[t], z[t]], \tag{6}$$

encounters many difficulties ([1–18]). One of the ways of overcoming these difficulties ([3]) is connected with introducing, among the arguments of the function u, the realized values of the quantity v. In the present note another form of regularization of the problem is discussed, connected with weakening the meeting condition (5), described in works ([15,18]).

Let, for a given $\varepsilon > 0$, for the initial state $y[t_0], z[t_0]$ and for the given realizations $u[t]$ and $v[t]$, the quantity $T^{\varepsilon}_{u,v}$ be defined as the least nonnegative number T satisfying the condition

$$\|\{y[t_0 + T] - z[t_0 + T]\}_{(m)}\| \leq \varepsilon. \tag{7}$$

We shall consider piecewise-constant realizations of the control u of the form

$$u[t] = c_k \quad \text{for } \tau_k \leq t < \tau_{k+1}, \tag{8}$$

where τ_k is the chosen sequence of numbers, $\tau_{k+1} - \tau_k = \delta > 0$. We shall say that a certain method of forming the control u, giving realizations (8), ensures that the motion $y[t]$ converges to the motion $z[t]$ no later than in time $T = \gamma_u$, if

$$\gamma_u = \sup_{\varepsilon > 0} \left\{ \limsup_{\delta \to 0} \left[\sup_v T^{\varepsilon}_{u,v} \right] \right\} < \infty. \tag{9}$$

In regular cases the time T_0 until the guaranteed encounter (5) of motion (1) with motion (2) is determined by the instant ϑ_0 of absorption [16] of process (2) by process (1), when the attainability region $G^{(2)}$ of motion (2) first lies in the attainability region $G^{(1)}$ of motion (1). The control $u = u[t]$ is then constructed in the form of an extremal control attracting the motion $y[t]$ to the point x^0, where the boundaries of the regions $G^{(1)}$ and $G^{(2)}$ are tangent. In the less regular cases considered by us, when the index (9) is used, the instant ϑ_ε of ε-absorption participates in the matter in an analogous way. This quantity is defined as follows. Suppose that from the given initial state $y[t_0]$, with $u(t) \equiv 0$, the motion $y(t)$ at the instant $t = \vartheta$ is brought to the state $y(\vartheta) = x$. We shall call the set of all those points y for which one can ensure the inequality

$$\left\| \{y(\vartheta) - y\}_{(m)} \right\| \leq \varepsilon \tag{10}$$

by choosing a programmed control $u(t)$ $(t_0 \leq t \leq \vartheta)$, constrained by the condition $\|u(t)\| \leq \mu$, the ε-attainability region and denote it by the symbol $G_\varepsilon^{(1)}[\vartheta - t_0, x]$. The region $G_\varepsilon^{(2)}[\vartheta - t_0, x]$ is defined in an analogous way. The instant $t = \vartheta$ at which the region $G_0^{(2)}[\vartheta - t_0, z(\vartheta)]$ first lies in the region $G_\varepsilon^{(1)}[\vartheta - t_0, y(\vartheta)]$ will be called the instant of ε-absorption. In what follows we shall need the notion of regular absorption of process (2) by process (1) for $0 \leq t \leq \vartheta$. Regularity of ε-absorption for $0 \leq t \leq \vartheta$ is based on the following two conditions (for every sufficiently small $\eta > 0$):

I. For all y, z and $\eta \leq \tau \leq \vartheta$ for which the region $G_0^{(2)}[\tau, z]$ is contained in the region $G_\varepsilon^{(1)}[\tau, y]$, the boundaries of these regions are tangent at no more than one point x^ε.

II. In a neighborhood of the point x^ε, the boundary of the region $G_0^{(2)}$ has greater curvature than the curvature of the boundary of the region $G_\varepsilon^{(1)}$.

Let x^0 be the point of the region $G_0^{(1)}$ closest to the point x^ε, and let $x_\varphi^{(1)}$ be the point of intersection with the boundary of $G_\varepsilon^{(1)}$ of the ray drawn from the point x^0 and making an angle φ with the vector $x^\varepsilon - x^0$. Then the condition

$$\left\|x_\varphi^{(1)} - x_\varphi^{(2)}\right\| \geq \beta\varphi^2 \quad (|\varphi| < \alpha),$$

must be satisfied, where α, β are positive constants (depending, perhaps, on η); $x_\varphi^{(2)}$ is the point of intersection of the ray φ with the boundary $C_0^{(2)}$ (if there is no such point, we regard the condition as satisfied).

We shall call the absorption of process (2) by process (1) regular for $0 \leq t \leq \vartheta$ if, for $0 \leq t \leq \vartheta$, the ε-absorptions are regular for all sufficiently small $\varepsilon > 0$.

The problem of programmed control of a linear system is a problem of moments [19]. Hence it follows [20] that all points w constituting the regions $G_\varepsilon^{(i)}[\tau, x]$, respectively, are obtained from the inequalities

$$\varepsilon\|l\| + \mu(v)\int_0^\tau \left\|\{F^{(i)}[\tau - t]B^{(i)}\}_{(m)}^* l\right\| dt - (w - x)^* l \geq 0, \tag{11}$$

which must be satisfied for every vector l. Here $F^{(i)}[t]$ is the fundamental matrix of solutions of equation (i) $(i = 1, 2)$. From the inequality

from (11) an effective notation for the conditions of regular absorption of (2) by process (1) is derived. We note that inequality (11) is most convenient to use for recording regularity conditions of absorption analogous to condition II, but expressed in terms of the angle φ that the supporting vector l (i.e., the vector l normal to the supporting hyperplane for the domain $G_\varepsilon^{(1)}$) makes with the supporting vector l^0 to $G_\varepsilon^{(1)}$, reconstructed at the point x^ε. Then from (11) there follows an effective form of writing the condition of regular absorption in the form of the inequality $\rho[l, l^0] \geq \beta\varphi^2$, where $\rho[l, l^0]$ is the distance, computed from (11), from the hyperplane $H_l^{(1)}$, supporting $G_\varepsilon^{(1)}$, to the domain $G_0^{(2)}$. (Incidentally, the effectiveness of such a notation is relative, since computational difficulties arise in concrete cases. The exception is perhaps only the case of objects of the same type [18] and some other analogous cases.)

Let us finally define the quantities $\vartheta[\tau_k]$. For $\tau_0 = t_0$, as $\vartheta[\tau_0]$ one chooses the moment of absorption of process (2) by process (1). Suppose the quantity $\vartheta[\tau_k]$ has been realized. We find, for the initial state $y[\tau_{k+1}], z[\tau_{k+1}]$, the moment of absorption ϑ_{k+1}. If $\vartheta_{k+1} \leq \vartheta[\tau_k]$, then $\vartheta[\tau_{k+1}] = \vartheta_{k+1}$. If, however, $\vartheta_{k+1} > \vartheta[\tau_k]$, then $\vartheta[\tau_k]$ is the moment of ϑ_ε ε-absorption, for some $\varepsilon > 0$; then we put $\vartheta[\tau_{k+1}] = \vartheta[\tau_k]$.

The following assertion is valid: *If the initial state $y[t_0], z[t_0]$ is such that $\vartheta[t_0] < \infty$, and if for $0 \leq \tau \leq \vartheta[t_0] - t_0$ process (1) absorbs process (2) regularly, then one can construct a control*

$$u[t] = u[y[\tau_k], z[\tau_k], \vartheta[\tau_k]], \tag{12}$$

which ensures that motion (1) is brought together with motion (2) no later than within the time $T = \vartheta[t_0] - t_0$.

The given control u, solving the problem of bringing together, is constructed as an extremal control which at the moment $t = \tau_k$ aims the motion $y[t]$ at the point x^0 nearest to the point x^ε, where the boundaries of the domains $G_\varepsilon^{(2)}[\vartheta[\tau_k], y(\vartheta[\tau_k])]$ and $G_0^{(2)}[\vartheta[\tau_k], z(\vartheta[\tau_k])]$ touch.

This control $u[t]$, for $t = \tau_k$, is determined from the maximum condition

$$h^*[t]u[t] = \nu^*\left(F^{(i)}[\vartheta_\varepsilon t]B^{(i)}\right)_{\{m\}}u[t] = \max_u \quad \text{for } \|u\| \leq \mu.$$

At the same time it is sometimes also expedient to regularize the control $u[\tau_k]$ by averaging $u[t]$ over $\tau_k \leq t < \tau_{k+1}$.

In conclusion we note that the regularization of the meeting problem under consideration is naturally coarser than the regularization of this same problem using the value v in the formation of the control u. However, under the regularization described by us, the motion $z[t]$ is no longer discriminated quite so explicitly.

We also note that the described method of forming the control u in a number of typical cases ensures the minimax of the time T until the objects come together. Namely, suppose it is known that for any control $u[y[t], z[t]]$ there exists a realization $v[t]$ which ensures the inequality

$$\|\{y[t] - z[t]\}_{\{m\}}\| > \varkappa[\varepsilon] > 0$$

for $t_0 \le t \le \vartheta_0[t] - \varepsilon$ ($\varepsilon > 0$). Then the above-described method of control ensures the natural minimax

$$\min_u \gamma_u = \vartheta_0[t_0] - t_0.$$

Such a situation occurs, in particular, in the case of linear objects of the same type.

Finally, it should be said that the regularization described in this note, associated with freezing the moment of absorption ϑ_ε, proves useful also in sufficiently regular cases when, however, the roots of the equation determining ϑ_ε are unstable with respect to small changes in the position $y[\tau], z[\tau]$ of the system.

An example of such a case may be the problem (moreover, even a sufficiently regular one) of the meeting of free material points $m^{(1)}$ and $m^{(2)}$, controlled by forces of arbitrary direction, but constrained by the condition $\|u\| \le \mu$, $\|v\| \le \nu$, with $\mu/m^{(1)} - \nu/m^{(2)} = 1$, and with the initial data as follows: the initial coordinates of the first point are $\xi_1^{(1)} = y_1 = -3\sqrt{2}/4$, $\xi_2^{(1)} = y_2 = \sqrt{2}/4$, the initial velocities $\dot{\xi}_1^{(1)} = y_3 = \sqrt{2}$, $\dot{\xi}_2^{(1)} = y_4 = 0$; the initial coordinates and velocities of the second point $\{\xi_i^{(2)}, \dot{\xi}_i^{(2)}\} = \{z_1, \dots, z_4\}$ are zero. The meeting is realized only in the coordinates ($\xi_i^{(1)} = \xi_i^{(2)}$ $(i = 1, 2)$).

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Note: Figure translations are in progress. See original paper for figures.