ON INFINITE-DIMENSIONAL SIMPLE GRADED LIE ALGEBRAS

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A Lie algebra U is called **graded** if $U = \sum_{-\infty}^{\infty} U_i$ (direct sum), where the U_i are finite-dimensional subspaces such that $[U_i, U_j] \subset U_{i+j}$. In the present note the ground field is assumed to be the field of complex numbers, but the greater part of the results is valid for an arbitrary field of characteristic zero.

We shall consider simple graded algebras satisfying the additional condition

$$[U_{-1}, U_{-k}] = U_{-(1+k)} \quad (k > 0). \tag{1}$$

First of all we give the construction of an infinite-dimensional simple graded algebra $\widetilde{U}$, which is in a certain sense "universal" (see Theorem 2). In the algebra $\widetilde{U}$ the subalgebra $\widetilde{U}_- = \sum_{i=1}^{-\infty} \widetilde{U}_i$ is the free Lie algebra generated by the n-dimensional vector space $E_n = \widetilde{U}_{-1}$, with its natural grading. In other words, the elements of $\widetilde{U}_{-k}$ are linear combinations of elements $a_1 \circ a_2 \circ \cdots \circ a_k$, where $\circ$ is the operation in the free Lie algebra and $a_i \in E_n$. Further, by definition, for $k \geq 0$ the space $\widetilde{U}_k$ is $\mathrm{Hom}(E_n, \widetilde{U}_{k-1})$, and $\widetilde{U}_0 = \mathrm{Hom}(E_n, E_n)$, i.e. the space $\widetilde{U}_k$ is the space of all $(k+1)$-linear operators $A(x_1, x_2, \ldots, x_{k+1})$ acting in E_n with values in E_n.

We complete the definition of the commutation operation on the space $\widetilde{U}$ obtained as follows. Let $A_k = A_k(x_1, x_2, \ldots, x_k) \in \widetilde{U}_{k-1}$ and $A_l = A_l(x_1, x_2, \ldots, x_l) \in \widetilde{U}_{l-1}$. Then

$$[A_k, A_l] = A_k \square A_l - A_l \square A_k, \tag{2}$$

where

$$A_k \square A_l(x_1, x_2, \ldots, x_{k+l-1}) =$$

$$= \sum_{s=0}^{k-1} \sum_{i_1<i_2<\dots<i_s}^{l+s-1} A_k(x_{i_1}, x_{i_2}, \dots, x_{i_s}, A_l(x_1, x_2, \dots, \hat{x}_{i_1}, \dots, \hat{x}_{i_s}, \dots, x_{l+s}),$$

$$x_{l+s+1}, \dots, x_{k+l-1}); \tag{3}$$

here $\sum_{i_1<i_2<\dots<i_s}^{l+s-1}$ denotes summation over all ordered sets of s indices, each of which is not greater than $l+s-1$, and the symbol $\hat{x}_{i_p}$ means that in the sequence $x_1, x_2, \dots$ the term with number i_p is omitted.

To define the commutation operation between $\widetilde{U}_{-k}$ and $\widetilde{U}_l$, note that every element of the free Lie algebra $a_1 \circ a_2 \circ \dots \circ a_k$ can be written as a sum of elements of the form $a_{i_1} a_{i_2} \cdots a_{i_k}$, belonging to the free associative algebra generated by the space E_n (for example, $a_1 \circ a_2 = a_1 a_2 + a_2 a_1$). We define the commutation operation between an $(l+1)$-linear operator $A(x_1, x_2, \dots, x_{l+1})$ and an associative element of the k-th

of order $b \equiv b_1 b_2 \dots b_k$ by the formula

$$[A, b] = \begin{cases} A(b_1, b_2, \dots, b_k, x_1, x_2, \dots, x_{l-k+1}), & \text{if } l+1 \geq k, \\ A(b_1, b_2, \dots, b_{l+1}) \circ b_{l+2} \circ \dots \circ b_k, & \text{if } l+1 < k. \end{cases} \tag{4}$$

Thus the commutation operation in the algebra $\widetilde{U}$ is completely determined.

Theorem 1. *The algebra $\widetilde{U}$ is a simple Lie algebra.*

Theorem 2. *Let U be a graded algebra satisfying condition (1), containing no ideals belonging to $U_+ = \sum_{i=0}^{\infty} U_i$. Let $\widetilde{U}$ be the Lie algebra defined above, for which $\dim \widetilde{U}_{-1} = \dim U_{-1}$. Then there exist: 1) a graded subalgebra U^* of the algebra $\widetilde{U}$, containing $\widetilde{U}_-$; 2) a graded ideal $D \subset \sum_{i=-2}^{-\infty} \widetilde{U}_i$ of the subalgebra U^* such that $U^*/D = U$.*

The proof of Theorem 2 follows from the following considerations.

Lemma 1. *Let M be an arbitrary Lie algebra. Denote*

$$P_a^k(x_1, x_2, \dots, x_k) \equiv [\dots[[a, x_1]x_2], \dots, x_k],$$

where $a, x_i \in M$. Then

$$P_{[a,b]}^{k+l-1} = P_a^k \square P_b^l - P_b^l \square P_a^k, \tag{5}$$

where the operation $\square$ is defined by formula (3).

Lemma 2. *Let U be a graded algebra satisfying condition (1), containing no ideals belonging to $U_+ = \sum_{i=0}^{\infty} U_i$. Assign to each $a \in U_k$ $(k \geq 0)$ the $k+1$-linear operator $F(a)$ on U_{-1}:*

$$a \xrightarrow{F} [\dots[[a, x_1], x_2], \dots, x_{k+1}].$$

Under the mapping F, the subalgebra $U_+ = \sum_{i=0}^{\infty} U_i$ is isomorphically embedded in the subalgebra $\widetilde{U}_+ = \sum_{i=0}^{\infty} \widetilde{U}_i$.

We now denote by U_+^F the image of the algebra U_+ under the mapping F, and by D the graded ideal of the subalgebra $\widetilde{U}_-$ such that $U_- = \widetilde{U}_-/D$. It turns out that the space $\widetilde{U}_- + U_+^F$ is a subalgebra U^* of the algebra $\widetilde{U}$, and the ideal D is carried into itself by elements from U_+^F. This proves Theorem 2.

Let us give examples illustrating Theorem 2. In these examples the subalgebra U^* coincides with $N(D)$, the normalizer of D in the algebra $\widetilde{U}$.

The Lie algebra of all infinitesimal analytic transformations represented by finite series* is obtained in this way if, as the subspace D, one takes $\sum_{i=-2}^{-\infty} \widetilde{U}_i$. In this case the subspaces U_i $(i \geq 0)$ consist of all $i+1$-linear symmetric operators.

The finite-dimensional simple Lie algebra of the series B_n arises if, as the subspace D, one takes $\sum_{i=-3}^{-\infty} \widetilde{U}_i$. In this case $U_i = 0$ for $i \geq 3$, U_0 consists of all linear operators, U_1 of all bilinear operators of the form $\varphi(y)x$, where $\varphi(y)$ is an arbitrary linear function, and, finally, U_2 of all trilinear operators of the form $f(x,z)y$, where $f(x,z)$ is an arbitrary bilinear skew-symmetric function.

In order to obtain a Lie algebra isomorphic to the infinite-dimensional simple Lie algebra of all infinitesimal transformations of a $(2n+1)$-dimensional space preserving a linear differential form

* The latter restriction arises because the elements of a graded algebra, by definition, are finite sums of vectors from the subspaces U_i.

$$dy + \sum_{i=1}^{n} (x_i dx_i' - x_i' dx_i)$$

and represented by finite series (see the footnote on p. 37), one must take as $\widehat{U}_{-1}$ the $2n$-dimensional space with the given nondegenerate skew-symmetric form (x,y), and as the subspace D the subspace

$$P + \sum_{i=-3}^{-\infty} \widehat{U}_i,$$

where $P \subset \widehat{U}_{-2}$ consists of the elements $a \circ b$ for which $(a,b) = 0$. In this case the subspaces U_i $(i \geq 0)$ consist of all symmetric $(i+1)$-linear operators satisfying the condition

$$(A(x_1, \dots, x_i, z), y) + (z, A(x_1, x_2, \dots, x_i, y)) = 0,$$

and of all operators of the form

$$\sum_{l>k}^{i+1} (x_k, x_l) B_{i-1}(x_1, \dots, \hat{x}_k, \dots, \hat{x}_l, \dots, x_{i+1}) + \sum_{k-1}^{i+1} F(x_1, \dots, \hat{x}_k, \dots, x_{i+1}) x_k,$$

where B_{i-1} $(i \geq 2)$ is an operator of the same type as A, and

$$F(x_1, \dots, x_i) = (B_{i-1}(x_1, \dots, x_{i-1}), x_i);$$

moreover, for $i = 0, 1, 2$ there are added respectively the following operators: the identity operator E, all bilinear operators of the form

$$(x, y)a - (x, a)y - (y, a)x$$

and the trilinear operator

$$(y, z)x - (y, x) - (z, x)y.$$

We shall henceforth restrict ourselves to graded Lie algebras for which, along with (1), the analogous condition

$$[U_1, U_k] = U_{1+k} \quad (k > 0). \tag{6}$$

is satisfied.

With every linear Lie algebra G_0 one can associate a certain special graded Lie algebra G satisfying conditions (1) and (6). To construct the algebra G, take for G_1 the space generated by linear combinations of bilinear operators of the form $f(x)A(y)$, where $f(x)$ is a linear function on U_{-1} and $A \in G_0$. Next put $G_{k+1} = [G_1, G_k]$, where the commutator is defined by formula (3); then

$$G^* = U_- + G_0 + G_1 + \cdots$$

is a graded subalgebra of the algebra $\widehat{U}$. Consider the ideal D of the subalgebra $\widehat{U}_-$, generated by the elements a for which $[a, G_1] = 0$. This ideal D is an ideal of the algebra G^*, and

$$G = G^*/D.$$

Theorem 3. The algebra G is an algebra without graded ideals. All other graded algebras with the given linear subalgebra G_0, satisfying conditions (1) and (6) and containing no graded ideals, are obtained from the algebra G in the following way. To each subspace $G_1' \subset G_1$ invariant with respect to G_0, for which $[U_{-1}, G_1'] = G_0$, assign the algebra

$$G = G_-' + G_+',$$

where G'_+ is the subalgebra of the algebra G_+ generated by G'_1, and

$$G'_- = \widehat{U}_-/D';$$

here D' denotes the ideal generated by the elements

$$\{a \in \widehat{U}_- : [a, G'_1] = 0\}.$$

Let us note that if condition (6) is discarded, then there are considerably more graded algebras with the given G_0 which contain no graded ideals. In this case, instead of the algebra G, one should consider the graded algebra

$$\widehat{G} = \widehat{G}_- + \widehat{G}_+,$$

where

$$\widehat{G}_1 = G_1, \qquad \widehat{G}_k = \mathrm{Hom}(U_{-1}, \widehat{G}_{k-1})$$

and

$$\widehat{G}_- = \widehat{U}_-/D,$$

where D is the ideal of $\widehat{U}_-$ generated by the set

$$\{a \in \widehat{U}_- : [a, \widehat{G}_k] = 0,\ k = 1, 2, \ldots\}.$$

Theorem 4. If among the elements of U_0 of the graded algebra U there is an element a such that $[a, x] = x$ for all $x \in U_{-1}$, then every ideal of the algebra U is graded.

Corollary. For every linear Lie algebra G_0 containing the identity operator, there exists at most one graded simple algebra for which $U_0 = G_0$.

We now note one special case of Theorem 3. Let U_{-1} be an n-dimensional space with basis $e_1, e_2, \ldots, e_n$. As G_0 take the commutative linear algebra consisting of all transformations of the form

$$\sum_{i=1}^{n} x_i e_i \to \sum_{i=1}^{n} a_i x_i e_i,$$

and as G'_1 take the subspace of bilinear operators

$$x^i \left(\sum_{j=1}^{n} a_{ij} y_j e_j \right),$$

where a_{ij} are prescribed numbers ($i = 1, 2, \ldots, n$). The matrix a_{ij}, in which the elements of each row are given up to a common factor, is called in the theory of finite-dimensional simple Lie algebras the Cartan matrix. From Theorems 3 and 4 it follows that

Theorem 5. *The graded Lie algebra determined by the Cartan matrix (a_{ij}) is simple if and only if the matrix (a_{ij}) is nonsingular and has no invariant subspaces with a basis consisting of coordinate vectors.*

In conclusion we make two remarks. Every simple graded Lie algebra satisfying conditions (1) and (6) is uniquely determined by the triple of subspaces U_{-1}, U_0, U_1 with the prescribed commutation operations $[U_{-1}, U_1]$, $[U_0, U_{-1}]$, $[U_0, U_1]$. If, however, one restricts oneself only to condition (1), then there may exist infinitely many simple algebras with one and the same triple U_{-1}, U_0, U_1. An example is the series of simple graded algebras for which (see Theorem 2) the subalgebra U_- is a free Lie algebra, the subspaces U_i $(i = 0, 1, 2, \dots, k-1)$ consist of all $(i+1)$-linear symmetric operators, and the subspaces U_i $(i \geq k)$ consist of all $(i+1)$-linear operators symmetric in the last k arguments.

The exceptional simple Lie algebras may be represented as elements of infinite series of simple graded algebras. For example, if by $\widehat{C}_n$ one denotes the graded algebra with n-dimensional subspace U_{-1} which arises (see Theorem 2) if, as the subspace D, one takes the ideal generated by elements of the form $b \circ a \circ d \circ a$, and as U_+ the subalgebra of the algebra U_+ consisting of elements carrying the subspace D into itself, then for $n = 2$ one obtains the exceptional Lie algebra G_2, and for $n > 3$ infinite-dimensional simple graded Lie algebras.

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Note: Figure translations are in progress. See original paper for figures.