STUDY OF THE ANALYTIC PROPERTIES OF THE RESOLVENT KERNEL OF AN ORDINARY DIFFERENTIAL OPERATOR OF FOURTH ORDER ON A RIEMANN SURFACE

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Abstract

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STUDY OF THE ANALYTIC PROPERTIES OF THE RESOLVENT KERNEL OF AN ORDINARY DIFFERENTIAL OPERATOR OF FOURTH ORDER ON A RIEMANN SURFACE

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1°. Consider in the Hilbert space $L^2(0,\infty)$ the self-adjoint operator H, generated by the differential expression

$$Lf \equiv d^4f/dx^4 + q(x)f \tag{1}$$

and by the boundary conditions

$$f(0) = f'(0) = 0. \tag{2}$$

In what follows we shall assume that the function $q(x)$ is real-valued, infinitely differentiable on the interval $[0,a]$, $0 < a < \infty$, and vanishes for $x > a$, while $q(a) \neq 0$. For definiteness we shall assume that $q(a) > 0$. Let $s = \sigma + i\tau$ be a complex number; denote by $y_k(x,s)$ $(k = 1,2,3,4)$ the solutions of the homogeneous equation

$$Lf = s^4 f, \tag{3}$$

satisfying the initial conditions

$$\begin{aligned} &y_1^{(j)}(a,s) = (is)^j; \qquad y_2^{(j)}(a,s) = (-s)^j; \qquad j = 0,1,2,3;\\ &y_3(0,s) = y_3'(0,s) = y_3''(0,s) = 0; \qquad y_3'''(0,s) = 1;\\ &y_4(0,s) = y_4'(0,s) = y_4''(0,s) = 0; \qquad y_4'''(0,s) = 1. \end{aligned} \tag{4}$$

Denote the Wronskian of the solutions $y_k(x,s)$ $(k = 1,2,3,4)$ by $\delta(s)$. It is easy to verify that the resolvent R_s of the operator H, for all values of the parameter

$s = \sigma + i\tau$ $(\tau > 0,\ \sigma > 0)$ for which $\delta(s) \neq 0$, is an integral operator. Denote its kernel by $R(x, t, s)$.

By virtue of the initial conditions (4),

$$\delta(s) = y_1(0, s) y_2'(0, s) - y_2(0, s) y_1'(0, s). \tag{5}$$

Along with the operator H, consider the boundary-value problem H_0, generated in the space $C(0, a)$ by the differential expression (1) and by the boundary conditions

$$f(0) = f'(0) = 0; \tag{6}$$

$$(i+1) f''(a, s) - s\big(s(i-1) f(a, s) - 2 f'(a, s)\big) = 0, \tag{7}$$

$$(i+1) f'''(a, s) + s^2\big(2is f(a, s) + (i-1) f'(a, s)\big) = 0.$$

The aim of the present work is to establish a connection between the self-adjoint boundary-value problem defined in the space $L_2(0, \infty)$ by the fourth-order differential expression (1) and the boundary conditions (2), and the boundary-value problem defined in $C(0, a)$ by means of the same differential expression (1) and the boundary conditions (6)–(7), depending on the spectral parameter s.

For the case of a second-order equation, an analogous problem was considered in the works of T. Regge ([2,3]) and A. O. Kravitskii ([4]).

Theorem*. The following assertions are true:

1. The spectrum of the problem H_0 consists of a countable number of eigenvalues s_n, which, except possibly for a finite number, are simple;
2. The kernel $R(x, t, s)$ of the resolvent of the operator H, for fixed values $x \geq 0$ and $t \geq 0$, considered as a function of the spectral parameter s, is a meromorphic function whose poles coincide with the numbers s_n;
3. Any four infinitely differentiable functions $f_j(x)$ $(j = 0, 1, 2, 3)$ on the interval $[0, a]$, vanishing at the points $x = 0$ and $x = a$ together with all their derivatives, admit an expansion in uniformly convergent series in the eigenfunctions $\varphi_n(x)$ of the problem H_0

$$f_j(x) = \sum_{n=1}^{\infty} C_n s_n^j \varphi_n(x), \tag{8}$$

where the coefficients C_n, in the case of simple numbers s_n, are determined by the formula

$$C_n = \frac{\displaystyle\int_0^a \left(s_n^3 f_0(x) + s_n^2 f_1(x) + s_n f_2(x) + f_3(x)\right) \varphi_n(x)\, dx}{4s_n^3 \displaystyle\int_0^a \varphi_n^2(x)\, dx + 3(i+1)s_n^2\varphi_n^2(a) + is_n\varphi_n(a)\varphi_n'(a) - (1-i)\left(\varphi_n'(a)\right)^2}. \tag{9}$$

4. The expansion (8) is unique.

2°. We proceed to the proof of the formulated theorem. The proof of the theorem is based on several lemmas.

Lemma 1. The resolvent R_s^0 of the problem H_0 has the form

$$R_s^0 g(x) = \int_0^a R(x,t,s) g(t)\, dt. \tag{10}$$

As is known ([1]), the entire complex s-plane can be divided into 8 sectors by means of the bisectors of the coordinate angles in such a way that, for each sector T_l, the different fourth roots of unity ω_k can be ordered so that, for $s \in T_l$,

$$\operatorname{Re} s\omega_1 \le \operatorname{Re} s\omega_2 \le \operatorname{Re} s\omega_3 \le \operatorname{Re} s\omega_4. \tag{11}$$

Lemma 2. Let $s \in T_1$ and let ω_k $(k = 1,2,3,4)$ satisfy the inequalities (11); then there exist 4 linearly independent solutions of equation (3), $\varphi_k(x,s)$, $k = 1,2,3,4$, regular for sufficiently large $|s|$ and such that, for $j = 0,1,2,3$:

1) uniformly for $0 \le x \le a$, the relations

$$\varphi_k^{(j)}(x,s) = s^j e^{s\omega_k x}\left(\omega_k^j + O(1/s^3)\right); \tag{12}$$

2) at the point $x = a$,

$$\varphi_k^{(j)}(a,s) = s^j e^{s\omega_k a}\left(\sum_{\alpha=1}^{k} \omega_k^j Q_{k\alpha} + O\left(\frac{1}{s^5}\right)\right), \tag{13}$$

where

$$Q_{k,k} = 1 - \frac{\omega_k}{4s^3}\int_0^a q(t)\, dt, \tag{14}$$

$$Q_{k,\alpha} = \frac{\omega_\alpha}{4s^3(\omega_\alpha - \omega_k)}\left(q(a) - e^{s(\omega_\alpha - \omega_k)}q(0)\right), \qquad k \ne \alpha. \tag{15}$$

* We note that the assertions of the theorem remain valid under weaker conditions on the smoothness of the functions $q(x)$ and $f_j(x)$. In addition, one may restrict the requirements of vanishing at zero for $x = 0$ and $x = a$ to only a finite number of derivatives of the functions $f_j(x)$.

The proof of the first part of the lemma is given in the monograph ([1]). The second assertion of the lemma is established analogously.

We now introduce into consideration the functions $z_j(x,s)$ $(j = 1,2,3,4)$, $s \in T_l$, which are solutions of equation (3) and satisfy the conditions $z_j^{(p)}(a,s) = (s\omega_j)^p, \quad p = 0,1,2,3;\ j = 1,2,3,4$.

Define the function

$$\delta_{j_1,j_2}(s) = z_{j_1}(0,s)z'_{j_2}(0,s) - z'_{j_1}(0,s)z_{j_2}(0,s). \tag{16}$$

From the definition of the functions $z_j(x,s)$ it follows that, depending on the choice of the sector T_l, the functions $y_1(x,s)$ and $y_2(x,s)$ will coincide with some pair of functions $z_{j_1}(x,s)$ and $z_{j_2}(x,s)$, so that $y_1(x,s) = z_{j_1}(x,s)$ and $y_2(x,s) = z_{j_2}(x,s)$, and the Wronskian $\delta(s) = \delta_{j_1 j_2}(s)$.

Lemma 3. Let the function $\delta_{j_1 j_2}(s)$ be defined by formula (16) and let s belong to one of the sectors T_j; then, for large $|s|$,

$$\begin{aligned}\delta_{j_1 j_2}(s) = e^{-sa(\omega_1+\omega_2)}(\omega_2-\omega_1)\begin{vmatrix} b_{j_{11}}(s) & b_{j_{21}}(s) \\ b_{j_{12}}(s) & b_{j_{22}}(s)\end{vmatrix}\left(1+O\left(\frac{1}{s^3}\right)\right) \\ + e^{-sa(\omega_1+\omega_3)}(\omega_3-\omega_1)\begin{vmatrix} b_{j_{11}}(s) & b_{j_{21}}(s) \\ b_{j_{13}}(s) & b_{j_{23}}(s)\end{vmatrix}\left(1+O\left(\frac{1}{s^3}\right)\right) + O(1),\end{aligned} \tag{17}$$

where

$$b_{jk}(s) = \begin{cases} O(1/s^5), & \text{for } j<k, \\ \prod\limits_{l\neq j} Q_{l,l}(s) + O\left(\frac{1}{s^5}\right), & \text{for } j=k, \\ -Q_{j,k}\prod\limits_{l\neq j;\, l\neq k} Q_{l,l}(s) + O\left(\frac{1}{s^5}\right), & \text{for } j>k. \end{cases} \tag{18}$$

We denote by T_1 the sector of the complex plane singled out by the conditions $s = \sigma + i\tau \in T_1$, if $\sigma > 0$, $\tau > 0$, $\sigma > \tau$; then, for $j = 1,2,3,\dots,8$,

$$T_j = e^{-i\frac{\pi}{4}(j-1)}T_1.$$

Lemma 4. The eigenvalues $s_k = \sigma_k + i\tau_k$ of the problem H_0 are arranged symmetrically with respect to the line $\sigma = \tau$. In the sectors T_2 and T_4, for large

$|s|$, the eigenvalues are simple and are determined by the following formulas: if $s_k \in T_2$, then

$$\sigma_k = \frac{k\pi}{a} + \frac{\pi}{4a} + o(1); \qquad \tau_k = -\frac{2}{a}\ln\frac{k\pi}{a} + \frac{1}{2a}\ln\frac{q(a)}{8} + o(1), \tag{19}$$

whereas if $s \in T_4$, then

$$\sigma_k = \frac{1}{2a}\ln\frac{q(a)}{8} - \frac{2}{a}\ln\frac{k\pi}{a} + o(1); \qquad \tau_k = -\frac{\pi k}{a} - \frac{s\pi}{4a} + o(1). \tag{20}$$

In the sectors T_1 and T_3 there can be only a finite number of eigenvalues of the problem H_0.

The first two assertions of the theorem follow from Lemmas 1–4.

3°. For the proof of the last two assertions of the theorem the following lemmas are used.

Lemma 5. Let the function $g(x)$ be infinitely differentiable and, at the points $x = 0$ and $x = a$, vanish together with all its derivatives; then the equality

$$R_s^0 g = -\frac{1}{s^4}\sum_{k=0}^{3}\frac{L^k g}{s^{4k}} + \frac{1}{s^{16}}R_s^0(L^4 g) \tag{21}$$

holds.

Let the functions $f_j(x)$, $j = 0, 1, 2, 3$, satisfy the conditions of the theorem.

Define the function

$$\Phi(x, s) = -\sum_{j=0}^{3} s^{3-j} R_s^0 f_j. \tag{22}$$

From formula (21) we find

$$\Phi(x, s) = \sum_{j=0}^{3}\frac{f_j(x)}{s^{j+1}} + \psi(x, s), \tag{23}$$

where

$$\psi(x, s) = \sum_{k=1}^{3}\sum_{j=0}^{3}\frac{L^k f_j}{s^{4k+j}} - \sum_{j=0}^{3}\frac{R_s^0(L^4 f_j)}{s^{12+j}}. \tag{24}$$

Denote by Γ_N the boundary of the octagon in the complex s-plane with vertices γ_j $(j = 1, \dots, 8)$, having coordinates $\gamma_1 = (R_N, R_N^\alpha)$; $\gamma_2 = (R_N, -R_N^\alpha)$; $\gamma_3 = (P_N^\alpha, -P_N)$; $\gamma_4 = (-P_N^\alpha, -P_N)$, where $0 < \alpha < 1$;

$$R_N = \frac{\pi}{2a}(2N+1) + \frac{\pi}{4a}, \qquad P_N = \frac{\pi}{2a}(2N+1) + \frac{5\pi}{4a}.$$

The remaining vertices are located symmetrically to these vertices with respect to the line $\sigma = \tau$.

Lemma 6. Let the function $\psi(x, s)$ be defined by formula (24) and let $N \to \infty$; then, uniformly in $x \in [0, a]$,

$$\lim \oint_{\Gamma_N} s^j \psi(x, s)\, ds = 0 \qquad (j = 0, 1, 2, 3).$$

Consider the integral

$$I_N = \frac{1}{2\pi i} \oint_{\Gamma_N} s^j \Phi(x, s)\, ds, \tag{25}$$

where the function $\Phi(x, s)$ is defined by formula (22). Passing to the limit as $N \to \infty$ and using the residue theorem, we obtain, uniformly in $x \in [0, a]$,

$$\lim I_N = \sum_l \operatorname{Res}\left(s^j \Phi(x, s)\right)_{s=s_l} \qquad (N \to \infty).$$

On the other hand, substituting expression (23) for $\Phi(x, s)$ in formula (25) and using Lemma 6, we obtain

$$\lim_{N\to\infty} I_N = f_j(x) = \sum_l \operatorname{Res}\left(s^j \Phi(x, s)\right)_{s=s_l}.$$

Computing the residues of the function $\Phi(x, s)$ at the points s_l, which are zeros of the function $\delta(s)$, we arrive at formulas (8) and (9).

The third assertion of the theorem is easily proved using the uniform convergence of the series (8). The method of proof of the theorem generalizes to the case of equations of higher orders, and also to the case of a complex-valued function $q(x)$.

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Note: Figure translations are in progress. See original paper for figures.