A STABILITY CRITERION FOR THE PROBABLE SPECTRUM OF LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS WITH RECURRENT COEFFICIENTS AND A CRITERION FOR THE ALMOST REDUCIBILITY OF SYSTEMS WITH ALMOST PERIODIC COEFFICIENTS

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Abstract

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A STABILITY CRITERION FOR THE PROBABLE SPECTRUM OF LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS WITH RECURRENT COEFFICIENTS AND A CRITERION FOR THE ALMOST REDUCIBILITY OF SYSTEMS WITH ALMOST PERIODIC COEFFICIENTS

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In papers (^{4,5}) the concept of the probable spectrum $\Lambda_p(A)$ of the linear system of differential equations

$$\dot{x} = A(t)x \tag{1}$$

($x \in E^n$, $A(t)$ is bounded and uniformly continuous on the line) was introduced and studied.

We shall say that the probable spectrum of system (1) is **stable** if, for every invariant measure μ on the dynamical system D_A (see (^{5})), there exists a set $M \subseteq R_A$, $\mu(M) = 1$, such that for every $\tilde{A}(t) \in M$ the system

$$\dot{x} = \tilde{A}(t)x \tag{2}$$

has stable exponents (see (^{6})) (R_A denotes the space of the dynamical system D_A).

In the present paper the following question is solved.

Suppose that $A(t)$ is recurrent (i.e., the dynamical system D_A is such that each of its trajectories is everywhere dense). Under what conditions is the probable spectrum of system (1) stable?

The results are then applied to the investigation of systems with almost periodic coefficients. In addition, at the end of the note an example is given of a system (1) with almost periodic coefficients that is not almost reducible (see (^{2}), § 21).

The following concept is fundamental for what follows.

Definition 1. We shall call system (1) **absolutely regular** if there exists a Perron transformation $x = U(t)u$ reducing it to triangular form

$$\dot{u} = P(t)u, \qquad P(t) = \begin{pmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ & \ddots & \\ 0 & & p_{nn}(t) \end{pmatrix}, \tag{3}$$

which satisfies the following conditions:

1) there exist

$$\lambda_i = \lim_{|t|\to\infty} \frac{1}{t} \int_0^t p_{ii}(\tau)\, d\tau \qquad (i = 1, 2, \dots, n); \tag{4}$$

2) for every $\varepsilon > 0$ there is a T such that the set $\Sigma_{\varepsilon,T}$ of those h for which, for at least one τ, $|\tau| > T$, the inequality

$$\left| \frac{1}{\tau} \int_0^\tau p_{ii}(h + \xi)\, d\xi - \lambda_i \right| \geq \varepsilon,$$

holds,

has relative measure on the line

$$\overline{\lim}_{t\to+\infty} \frac{1}{2t} \operatorname{mes} \Sigma_{\varepsilon,\tau} \cap [-t, t] < \varepsilon. \tag{5}$$

The following theorems hold, which show that, on the one hand, almost all systems (1) are absolutely regular and, on the other hand, these systems possess very special properties.

Theorem 1. *Almost every (in the sense of any invariant measure on* D_A*)* $\widetilde{A}(t) \in R_A$ *is such that system (2) is absolutely regular.*

Theorem 2. *Almost every (in the same sense as in Theorem 1) system (2) is reduced, by some Perron transformation* $x = U(t)u$*, to triangular form (3), satisfying conditions 1), 2) of Definition 1 and such that* $\lambda_i \geq \lambda_j$ *for* $i \leq j$*.*

For such a system (2) there exists a normal basis (see (^{2}), p. 28) of solutions

$$x_1(t), x_2(t), \dots, x_n(t),$$

having the properties:

$$1) \qquad \lim_{|t|\to\infty} \frac{1}{t} \ln \|x_i(t)\| = \lambda_i \quad (i = 1, 2, \dots, n); \tag{6}$$

2) For every $i = 1, 2, \dots, n$ we have: for every $\varepsilon > 0$ there exists a T such that the set $H_{\varepsilon,T}$ of those h for which, for at least one τ, $|\tau| > T$, for at least one solution $x(t)$ of system (1), which is a linear combination of those $x_k(t)$ for which $\lambda_k = \lambda_i$, the inequality

$$\left| \frac{1}{\tau} \ln \frac{\|x(h+\tau)\|}{\|x(h)\|} - \lambda_i \right| \geq \varepsilon,$$

has relative measure on the line

$$\overline{\lim}_{t \to +\infty} \frac{1}{2t} \operatorname{mes} H_{\varepsilon,T} \cap [-t, t] < \varepsilon.$$

I omit the proof of these theorems. With their help one proves

Theorem 3. *Let $A(t)$ be recurrent. Let the probable spectrum of system (1) be stable. Then system (1), by some Lyapunov transformation $x = L(t)y$ (such that $L(t)$ is uniformly continuous on the line), is reduced to triangular form (3), which satisfies the conditions:*

a) *if $\lambda_i = \lambda_j$, then $\lambda_k = \lambda_i$ for every k lying between i and j;*

b) *for each $i, j = 1, 2, \dots, n$ either*

$$\underline{\lambda}_{p_{ii}(t) - p_{jj}(t)} > 0,$$

or

$$\overline{\lambda}_{p_{ii}(t) - p_{jj}(t)} < 0,$$

or

$$\underline{\lambda}_{p_{ii}(t) - p_{jj}(t)} = \overline{\lambda}_{p_{ii}(t) - p_{jj}(t)} = 0,$$

where

$$\overline{\lambda}_{p(t)} = \lim_{t - \tau \to +\infty} \frac{1}{t - \tau} \int_\tau^t p(\xi)\, d\xi,$$

$$\underline{\lambda}_{p(t)} = \lim_{t - \tau \to +\infty} \frac{1}{t - \tau} \int_\tau^t p(\xi)\, d\xi$$

(i.e. system (1) satisfies the conditions of integral separatedness—closeness (see ([2]), Supplement § 17).

I omit the proof. (The converse theorem follows from Theorem 15.2.1 (2).)

Let a Lyapunov transformation $x = L(t)y$ be fixed ($L(t)$ is uniformly continuous on the line), reducing system (1) to triangular form (3), and suppose that requirements a) and b) of Theorem 3 are fulfilled. Define on

space R_A of the dynamical system D_A of the functions $\beta_i(\widetilde{A}(t))$ by the formula

$$\beta_i(\widetilde{A}(t)) = \lim_{k\to\infty} \dot{A}(t_k + t) = \lim_{k\to\infty} \sum_{\lambda_k=\lambda_i} p_{ii}(t_k + t) \tag{7}$$

$$(i = 1, 2, \dots, n)$$

(lim in (7) means not the limit, but any limit point of the sequence (in the sense of uniform convergence on intervals); thus, the functions $\beta_i(\widetilde{A}(t))$ are a priori multivalued).

Theorem 4. *Let $A(t)$ satisfy the conditions of Theorem 3. Let the functions $\beta_i(\widetilde{A}(t))$ be defined as indicated above.*

Then the functions $\beta_i(\widetilde{A}(t))$ $(i = 1, 2, \dots, n)$ are single-valued and continuous everywhere on R_A.

Let us now consider an important special case.

Theorem 5. *Let the probable spectrum of system (1) be stable and let the dynamical system D_A be strictly ergodic (see (1), p. 531). Then system (1) is almost reducible.*

We shall now apply the results obtained to the study of systems (1) with almost periodic coefficients.

It is easy to prove

Lemma 1. *Let $A(t)$ be almost periodic in t. In order that the exponents of system (1) be stable, it is necessary and sufficient that the exponents of every system (2) $(\widetilde{A}(t) \in R_A)$ be stable.*

From Lemma 1, Theorem 5, and Theorem 15.2.1 (2) it follows that

Theorem 6. *Let $A(t)$ be almost periodic in t. In order that system (1) be almost reducible, it is necessary and sufficient that its exponents be stable.*

Theorem 7. *There exists a system (1), not almost reducible (of any order $n \geq 2$), with almost periodic coefficients.*

This system $(n = 2)$ is constructed as follows: fix a series $\sum_{i=1}^{\infty} a_i$ such that

$$\sum_{i=1}^{\infty} |a_i| < +\infty,$$

and a sequence of natural numbers $m_i > 1$. Put

$$n_k = \prod_{i=1}^{k} m_i.$$

Put

$$A(t) = \sum_{i=0}^{\infty} A_i(t), \tag{8}$$

$$A_0(t) = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{for } sn_1 - 1 \le t < sn_1 \text{ (s any integer)}, \\ \frac{\pi}{2}|\sin \pi t| \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{for the remaining } t, \end{cases} \tag{9}$$

and for $k \geq 1$:

$$A_k(t) = \begin{cases} \frac{\pi}{2}|\sin \pi t| \begin{pmatrix} 0 & a_k \\ -a_k & 0 \end{pmatrix} & \text{for } sn_k - 1 \le t < sn_k \text{ (s any integer)}, \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{for the remaining } t. \end{cases} \tag{10}$$

It is easy to see that $A(t)$, defined by formulas (8)–(10), is almost periodic. The series $\sum_{i=1}^{\infty} a_i$ and the numbers m_i can be chosen so that the resulting

the system (1) obtained is not almost reducible, but the proof of the possibility of such a choice cannot be explained here for lack of space.

Remark. Almost reducible systems with almost periodic coefficients were studied by B. F. Bylov (see $(^3)$), who obtained for such systems a generalization of the Floquet-Lyapunov theorem. We also note that this result of B. F. Bylov can be derived from Theorem 4. (B. F. Bylov considered the case of nonmultiple characteristic exponents.)

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Note: Figure translations are in progress. See original paper for figures.

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