CRITERION FOR HYDROMAGNETIC STABILITY OF PLASMA IN THE NEIGHBORHOOD OF THE MAGNETIC AXIS

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Abstract

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CRITERION FOR HYDROMAGNETIC STABILITY OF PLASMA IN THE NEIGHBORHOOD OF THE MAGNETIC AXIS

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The general stability criterion for an arbitrary equilibrium plasma configuration has the form ($^{1-5}$)

$$\langle S/2-\mathbf{jB}|\nabla V|^{-2}\rangle^2-\langle \mathbf{B}^2|\nabla V|^{-2}\rangle\langle\Omega-\mathbf{j}^2|\nabla V|^{-2}\rangle\geqslant 0. \tag{1}$$

Here $\mathbf{B}$ is the magnetic field; $\mathbf{j}=\operatorname{rot}\mathbf{B}$; V is the current volume of the system of nested magnetic surfaces (m.s.) with magnetic axis (m.a.) $V=0$,

$$\langle f\rangle=\frac{d}{dV}\int f\,d\tau,\qquad \nabla p=[\mathbf{jB}],\qquad p'=I'\Phi'-J'\chi',$$

$$\Omega=I'\Phi''-J'\chi'',\qquad S=\chi'\Phi''-\Phi'\chi'',$$

$\Phi(V)$ and $J(V)$ are longitudinal, while $\chi(V)$ and $I(V)$ are transverse fluxes of the vectors $\mathbf{B}$ and $\mathbf{j}$.

In the natural (^{6,7}) surface coordinate system θ,ζ,V

$$j^i=\{I',\,J',\,0\},\qquad B^i=\{\chi',\,\Phi',\,0\},\qquad B_i=\{\varphi_\theta+J,\,\varphi_\zeta-I,\,\varphi_V\},$$

$$A_i=\{\Phi,\,-\chi,\,0\},$$

where φ and $\mathbf{A}$ are the scalar and vector potentials of the magnetic field, and the indices θ,ζ,V denote partial derivatives with respect to the corresponding coordinate.

In the neighborhood of the m.a. criterion (1) is transformed to the form

$$p'\{V'''/V' - p'\langle B^{-2}(1+\varphi_\vartheta^2|\nabla\Phi|^{-2})\rangle\} > 0, \tag{2}$$

where primes denote derivatives with respect to Φ. We introduce a straightening (^{7}) axial coordinate system r, ϑ, s, rotating together with the m.s. of the approximation quadratic in r with velocity $\delta'(s)$. The metric of such a coordinate system is determined by the quadratic form $dx^2 = g_{ik}dx^i dx^k$,

$$\begin{aligned} dx^2 = (q_0 + q_1\cos 2\vartheta)\,dr^2 + (q_0 - q_1\cos 2\vartheta)r^2 d\vartheta^2 + (h_s^2 + q_2 r^2)\,ds^2 \\ - 2q_1 r\sin 2\vartheta\,dr\,d\vartheta + r(q_0' + q_1'\cos 2\vartheta)\,dr\,ds \\ - (2u'B_0^{-1} + q_1'\sin 2\vartheta)r^2 d\vartheta\,ds, \end{aligned} \tag{3}$$

where

$$q_0(s) = B_0^{-1}\operatorname{ch}\eta; \qquad q_1(s) = -B_0^{-1}\operatorname{sh}\eta; \qquad q_2 = (u'\lambda_1\cos\vartheta - \lambda_2'\sin\vartheta)^2 +$$

$$+(u'\lambda_2\sin\vartheta + \lambda_1'\cos\vartheta)^2; \qquad \lambda_1 = (B_0 e^{\eta})^{-1/2}; \qquad \lambda_2 = (B_0 e^{-\eta})^{-1/2};$$

$$u' = \delta' - \varkappa; \qquad h_s = 1 - rf_c; \qquad f_c = c_1\cos\vartheta + c_2\sin\vartheta; \qquad c_1 = k\lambda_1\cos\theta; \qquad c_2 = k\lambda_2\sin\theta;$$

e^η is the ratio of the semiaxes of the normal elliptic cross-section of the m.s. in the neighborhood of the m.a.; $\delta(s)$ is the angle of the semiaxis of the ellipse with the principal normal to the m.a.; $k(s)$ and $\varkappa(s)$ are the curvature and torsion of the m.a.; $j_0(s)$ and $B_0(s)$ are the values of j and B on the m.a. The determinant g_{ik} is equal to $g = r^2 B_0^{-2} h_s^2$. We introduce the notations $f_a = a_1\cos\vartheta + a_2\sin\vartheta + a_3\cos 3\vartheta + a_4\sin 3\vartheta$, $F_b = b_0 + b_1\cos 2\vartheta + b_2\sin 2\vartheta + b_3\cos 4\vartheta + b_4\sin 4\vartheta$ for the corresponding trigonometric functions with coefficients $a_i(s)$, $b_i(s)$, etc.

We represent the contravariant components of $\mathbf{B}$ in the coordinates r, ϑ, s in the form

$$\begin{aligned} \sqrt{g}B^1 = r^3 f_a + r^4 F_l + \dots, \qquad \sqrt{g}B^2 = \nu' r + r^2 f_b + r^3 F_m + \dots, \\ \sqrt{g}B^3 = r + r^2 f_c + r^3 F_n + \dots. \end{aligned} \tag{4}$$

The expressions for the scalar potential φ and the surface function ψ will be

$$\varphi = \int_0^s B_0\,ds + r^2 F_D + r^3 f_\varphi + \dots, \qquad \psi = r^2 + r^3 f_\alpha + \dots. \tag{5}$$

We seek the transformation from the coordinates r, ϑ, s to the coordinates θ, ζ, V in the form

$$V = V_\psi(r^2 + r^3 f_\alpha + ...), \qquad \theta = k_0\vartheta + \int_0^s k_1\, ds + r f_\beta + ...\,;$$
$$\zeta = \int_0^s k_2\, ds + r f_\gamma + ...\,, \tag{6}$$

where for $k_i(s)$ and $\nu'(s)$ one obtains the formulas

$$k_0 = 1/2\pi, \qquad k_1 = \chi_V/B_0 - \nu'/2\pi, \qquad k_2 = \Phi_V/B_0,$$
$$\nu' \operatorname{ch}\eta = u' - j_0/2B_0, \tag{7}$$
$$\Phi_\psi = \pi, \qquad 2\pi\chi_\Phi = \oint \nu'\, ds, \qquad V_\Phi = \oint B_0^{-1}\, ds, \qquad J_\psi = \pi j_0/B_0, \qquad I_\psi = p_\psi V_\Phi + J_\psi \chi_\Phi,$$

where the subscripts ψ, Φ, V denote derivatives with respect to ψ, Φ, V, and for the functions $f_\alpha, f_\beta, f_\gamma$ the standard equations are obtained

$$\partial f_\alpha/\partial s + \nu' \partial f_\alpha/\partial\vartheta = -2 f_a,$$
$$\partial f_\beta/\partial s + \nu' \partial f_\beta/\partial\vartheta = -k_0 f_b - (2k_1 + k_0\nu') f_c,$$
$$\partial f_\gamma/\partial s + \nu' \partial f_\gamma/\partial\vartheta = -2k_2 f_c. \tag{8}$$

The coefficients D_i and n_0 are determined by the expressions $(D_3 = D_4 = 0)$

$$4D_0 = B_0(\operatorname{ch}\eta/B_0)', \qquad 4D_1 = -B_0(\operatorname{sh}\eta/B_0), \qquad 2D_2 = \nu' \operatorname{sh}\eta,$$
$$2B_0^2 n_0 = -2p_\psi - \nu' j_0 + 2B_0 u'(\nu' - u' \operatorname{ch}\eta) - (B_0' + B_0\eta')e^{-\eta}/4 -$$
$$-(B_0' - B_0\eta')e^{\eta}/4 + k^2 B_0(e^{-\eta}\cos^2\delta + e^{\eta}\sin^2\delta). \tag{9}$$

Calculation of the surface functions entering criterion (2) gives

$$\pi V'' = -\oint (n_0 - 2a_i c_i) B_0^{-1}\, ds, \qquad 2\pi^2\chi'' = -\oint \{(n_0 - a_i c_i)\nu' - m_0 + a_i b_i\}\, ds, \tag{10}$$
$$V'\langle B_0^{-2}\rangle = \oint B_0^{-3}\, ds, \qquad \langle \varphi_\theta^2 B_0^{-2} |\nabla\Phi|^{-2}\rangle = V' \oint (2\operatorname{ch}\eta/2)^{-1}(e^{\eta/2}\gamma_1^2 + e^{-\eta/2}\gamma_2^2)\, ds.$$

If these expressions are substituted into (2), then after integration by parts we obtain the following condition for plasma stability in a neighborhood of the magnetic axis:

$$-p_\psi \oint ds\{B_0^{-2}\operatorname{ch}\eta[2k^2(1-\varepsilon\cos 2\delta) - j_0^2 B_0^{-2}(1-\varepsilon^2) - 4{u'}^2 g_2 - \eta {a'}^2 -$$

$$-3B_0^{-2}B_0'^2 + 4B_0^{-1}B_0'\eta'\varepsilon] - 8kB_0^{-3/2}(\alpha_1 e^{-\eta/2}\cos\delta + \alpha_2 e^{\eta/2}\sin\delta) +$$

$$+2p_\psi V_\Phi^2 \operatorname{ch}^{-1}\eta/2\,(e^{\eta/2}\gamma_1^2 + e^{-\eta/2}\gamma_2^2)\} > 0. \qquad (11)$$

Here $\varepsilon = \operatorname{th}\eta$, and the term $S^2/4$ has been discarded.

Use of the equation $\operatorname{div}\mathbf{B}$ and the transformation rules $B_i^* = g_{ik}B^k$ leads, after elimination of the functions $\varphi_i(s)$, to the following equations

for $a_i(s)$.

$$\begin{aligned} a_1' + \nu' a_2 &= (2q_0 + q_1)^{-1}(P_1 - 2q_1 a_3), & a_3' + 3\nu' a_4 &= -2a_3, \\ a_2' - \nu' a_1 &= (2q_0 - q_1)^{-1}(P_2 - 2q_1 a_4); & a_4' - 3\nu' a_3 &= -2a_4. \end{aligned} \qquad (12)$$

The functions γ_1 and γ_2 satisfy analogous equations according to (8), and the right-hand sides of the equations for γ_1 and γ_2 are known, while the right-hand sides of the equations for a_1 and a_2 contain arbitrary functions $a_3(s)$ and $a_4(s)$, as well as γ_1 and γ_2:

$$2P_1 = (3q_0+q_1)(c_1' - \nu' c_2) - (q_0' + q_1')c_1 + 6B_0^{-1}u'c_2 + B_0^{-1}(J_\Phi c_2 + 2p_\psi V_\Phi \gamma_1), \qquad (13)$$

$$2P_2 = (3q_0 - q_1)(c_2' - \nu' c_1) - (q_0' + q_1')c_2 + 6B_0^{-1}u'c_1 - B_0^{-1}(J_\Phi c_1 - 2p_\psi V_\Phi \gamma_2).$$

The functions a_3 and a_4 can be determined if the profile of the cross section of the boundary magnetic surface Σ is specified in the coordinate system r_1, ϑ_1: $r_1\cos\vartheta_1 = x_0 + r\cos\vartheta$, $r_1\sin\vartheta_1 = y_0 + r\sin\vartheta$, whose axis is displaced by the distance $r_0 = \sqrt{x_0^2 + y_0^2}$ from the magnetic axis. In this coordinate system $\psi = (1 + r_1 f_\Delta)(r_1^2 + r_1^3 f_\sigma - \rho^2) + \text{const}$, where $\rho = \text{const}$, and the σ_i may be regarded as prescribed functions determining the shape of the cross section of the surface Σ. For small displacements $r_0 \ll \rho$, the ellipticity ε of the near-axis sections of the magnetic surfaces coincides with the ellipticity of the surface Σ, $\vartheta_1 \simeq \vartheta$, $\Delta_3 = \Delta_4 = 0$, $\Delta_1 = 2x_0/\rho^2$, $\Delta_2 = 2y_0/\rho^2$, $a_1 = \Delta_1 + \sigma_1$, $a_2 = \Delta_2 + \sigma_2$, $a_3 = \sigma_3$, $a_4 = \sigma_4$.

In particular, for an elliptical cross section of the surface Σ we have $a_3 = a_4 = 0$, $a_1 = \Delta_1$, $a_2 = \Delta_2$, so that the functions a_1 and a_2 turn out to be proportional to the displacements of the magnetic axis, $a_1 = 2x_0/\rho^2$, $a_2 = 2y_0/\rho^2$. It should be noted that the choice of the asymmetry parameters σ_i of the profile of the cross section of the surface Σ is restricted by the separatrix of the family of surfaces $\Psi(r_1, \vartheta_1) = r_1^2 + r_1^3 f_\sigma(\vartheta_1) = \text{const}$.

The solution of systems of equations of type (12) in integral form and in the form of series is given in (7).

The integrand in the criterion for plasma stability in the neighborhood of the magnetic axis (11) depends essentially on 9 parameters: on the curvature and torsion of the magnetic axis k and χ, on the ellipticity of the near-axis sections of the magnetic surfaces ε and the asymmetry parameters σ_i, on the rate of rotation of the sections $\delta'(s)$, on the variation of the longitudinal field $B_0(s)$, on the ratio $j_0/B_0 = \text{const}$, and on the pressure $p(V)$.

In the case of a straight magnetic axis $k = 0$, the plasma can be stable according to condition (11) with $p'(V) < 0$ only if $\varepsilon\eta' B_0' \neq 0$.

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