SOME PROPERTIES OF ALGEBRAS OF RECURSIVE FUNCTIONS

MATHEMATICS

1968

SovietRxiv

SovietRxiv

Abstract

Full Text

UDC 51.01:518.5

MATHEMATICS

E. A. POLYAKOV

SOME PROPERTIES OF ALGEBRAS OF RECURSIVE FUNCTIONS

(Presented by Academician P. S. Novikov on 31 III 1967)

1. Let $\mathfrak{A}_{\text{pr}} = \langle A_{\text{pr}}; +, *, i\rangle$, $\mathfrak{A}_{\text{or}} = \langle A_{\text{or}}; +, *, ^{-1}\rangle$, $\mathfrak{A}_{\text{chr}} = \langle A_{\text{chr}}; +, *, ^{-1}\rangle$ be, respectively, the algebras of one-place primitive-recursive functions (p.r.f.), general-recursive functions (g.r.f.), and partial-recursive functions (p.r.f.) ([1]). By a basis of an algebra we shall mean a system of generators (not necessarily independent) of this algebra. It is known that the algebras $\mathfrak{A}_{\text{pr}}, \mathfrak{A}_{\text{or}}, \mathfrak{A}_{\text{chr}}$ are finitely generated (the functions $S(x) = x + 1$, $q(x) = x \dot{-} [\sqrt{x}]^2$ form a basis of each of these algebras).

In the present note, for each of the indicated algebras, we establish the absence of an algorithm which, for every constructively given system of functions from this algebra, would determine whether or not this system of functions is a basis.

By χ we shall denote a certain principal numbering (([2]), §11) of the set A_{chr}. In particular, χ may be a "Gödel" numbering of A_{chr}. Let G be one of the sets $A_{\text{pr}}, A_{\text{or}}, A_{\text{chr}}$. Let F, G be sets of functions and $F \subseteq G \subseteq A_{\text{chr}}$.

The set F is called **recursive relative to** G if there exists a partial-recursive function f such that

$$f(n) = \begin{cases} 1, & \text{if } \chi n \in F, \\ 0, & \text{if } \chi n \in G - F, \end{cases}$$

where by χn is denoted the function whose number in the numbering χ is n.

Let us note that there exist nontrivial recursive subsets of the set G. For example, let $P(x_1, \dots, x_k)$ be some recursive predicate and $y_1, y_2, \dots, y_k$ arbitrary fixed natural numbers. Define the set of functions $F_P^{y_1,\dots,y_k}$ in the following way:

$$f(x) \in F_P^{y_1,\dots,y_k} \leftrightarrow f(x) \in G \ \&\ P(f(y_1), \dots, f(y_k)) = \text{T}.$$

It is not difficult to see that $F_P^{y_1,\dots,y_k}$ is a recursive subset of G. Let F be any set of arithmetical functions. We shall say that a function $f(x)$ **almost belongs** to F if there exist a function $g(x) \in F$ and a natural number m such that $(\forall x)(x \geq m \to f(x) = g(x))$.

Theorem 1 gives us a sufficient condition for nonrecursiveness.

Theorem 1. *Let G be one of the sets $A_{\text{pr}}, A_{\text{or}}, A_{\text{chr}}$, and let χ be a principal numbering of the set A_{chr}. Let $F \subseteq G$. If there exists a function $f(x) \in G - F$ which does not almost belong to F, then the set F is nonrecursive relative to G.*

In ([3]) it was noted that every basis of any of the algebras $\mathfrak{A}_{\text{pr}}, \mathfrak{A}_{\text{or}}, \mathfrak{A}_{\text{chr}}$ must contain a function g such that the equation $g(y) = x$ has an infinite number of solutions for infinitely many values of x. In ([3]) it was also shown that each of the algebras $\mathfrak{A}_{\text{pr}}, \mathfrak{A}_{\text{or}}, \mathfrak{A}_{\text{chr}}$ has one-element bases.

Let $\mathfrak{B}$ be one of the algebras $\mathfrak{A}_{\text{pr}}, \mathfrak{A}_{\text{or}}, \mathfrak{A}_{\text{chr}}$, and let B be the set of all its one-element bases. It is easy to see that there is a function belonging to the basic set of the algebra $\mathfrak{B}$ which is not almost contained in B. Hence, from Theorem 1 it follows that the set B is not recursive; and this shows the absence of the algorithm discussed at the beginning of the note.

From Theorem 1 there also follows a number of other consequences concerning algorithmic undecidability. For example, there are no algorithms which, for a given general recursive function, would determine whether it is a partial recursive function, whether it is a regular function (i.e., assumes all natural numbers as its values), or whether it is a bounded function.

2. S. C. Kleene ([4]) showed that for every general recursive function $f(x)$ there exist partial recursive functions ψ, g such that

$$f(x) = \psi(\mu y(g(y) = x)) = \psi(g^{-1}(x)), \tag{1}$$

where g assumes all natural numbers as values.

In the present note it is shown that there does not exist a partial recursive function ψ such that every general recursive function f can be represented in the form (1) for a suitable partial recursive function g, contrary to Kleene' s conjecture.

3. An arithmetical function $F(x)$ is called a **general-recursive extension** of a partial recursive function $f(x)$ if $F(x)$ is a general recursive function coinciding with f on the domain of definition of f. Some examples are known (([5]), §6) of partial recursive functions that do not have general-recursive extensions. The question arises: what necessary and sufficient conditions must a partial recursive function f satisfy in order that it can be extended to a general recursive function? Theorem 2 indicates such conditions for partial recursive functions that are inverses of suitable partial recursive functions.

Theorem 2. Let $f(x)$ be a partial recursive function. The function $f^{-1}(x)$ can then and only then be extended to a general recursive function when ρf (the range of values of f) is a recursive set, or $(\exists x)(f(x)$ is not defined).

Theorem 3 gives necessary and sufficient conditions for when a partial recursive function $f(x)$ can be represented in the form $f(x) = g^{-1}(x)$ for a suitable partial recursive function g.

Theorem 3. A partial recursive function f can be represented in the form $f(x) = g^{-1}(x)$ for some partial recursive function g if and only if [f is single-valued & $(0 \in \rho f \vee \rho f = \emptyset)$ & ρf is a recursive set].

Here the symbol $\emptyset$ denotes the empty set. Let us consider the algebra $\mathfrak{A}_{\text{or}}$. This algebra is partial, since the operation $^{-1}$ is applied only to regular functions.

Let $F(x)$ be some general recursive function. Extend the operation $^{-1}$ to the operation $^{\alpha}$ in the following way:

$$f^{\alpha} = \begin{cases} \mu y(f(y) = x), & \text{if } (\exists y)(f(y) = x), \\ F(x), & \text{otherwise.} \end{cases}$$

Theorem 2 excludes the possibility of such an extension of the operation $^{\alpha}$, since for some general recursive functions f the function f^{α} may fail to be general recursive.

4. Further results concern formal definability in the algebra $\mathfrak{A}_{\text{chr}}$.

A set of partial recursive functions A is called **formally definable in the algebra** $\mathfrak{A}_{\text{chr}}$ if there exists a formula of predicate calculus $\mathfrak{A}(x)$ with one free object variable, containing only the functional symbols $+, *, ^{\alpha}$, such that

$$f(x) \in A \leftrightarrow \mathfrak{A}(f) = I.$$

The following Theorem 4 establishes a connection between sets of partial recursive functions formally definable in $\mathfrak{A}_{\text{chr}}$ and arithmetical sets (([5]), §13). Let χ be a Gödel numbering of A_{chr}.

Theorem 4. A set of partial recursive functions A is formally definable in the algebra $\mathfrak{A}_{\text{pr}}$ if and only if A is the χ-image of a suitable arithmetical set of natural numbers.

From Theorem 4 it follows, in particular, that the sets of all p.r.f.' s and of all general recursive functions are formally definable in the algebra $\mathfrak{A}_{\text{pr}}$.

Ivanovo State
Pedagogical Institute
named after D. A. Furmanov

Received
31 III 1967

References Cited

[1] A. I. Mal' tsev, UMN, 16, No. 3, 3 (1961).
[2] V. A. Uspenskii, *Lectures on Computable Functions*, Moscow, 1960.
[3] E. A. Polyakov, *Algebra and Logic*, 3, No. 1, 41 (1964).
[4] Th. Skolem, Math. Scand., 1, No. 2, 213 (1953).
[5] A. I. Mal' tsev, *Algorithms and Recursive Functions*, Moscow, 1965.

Note: Figure translations are in progress. See original paper for figures.