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Abstract

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MATHEMATICS

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AN ANALOGUE OF THE PLANCHEREL FORMULA FOR HYPERBOLOIDS

(Presented by Academician P. S. Novikov on 23 I 1968)

1. Let G be a connected group of linear transformations of an n-dimensional real space preserving the quadratic form $[x,x] = -x_1^2 - \cdots - x_p^2 + x_{p+1}^2 + \cdots + x_n^2$ (a pseudo-orthogonal group), and let X be the hyperboloid $[x,x] = 1$. The measure $dx = |x_1|^{-1} dx_2 \cdots dx_n$ on X is invariant under the natural action $x \to xg$ of the group G. In the space $L^2(X)$ with respect to this measure there arises a unitary representation of the group G: $f(x) \to f(xg)$. For $q = n - p > 1$ the stationary subgroup of a point of the hyperboloid X is not compact; therefore harmonic analysis of functions on the hyperboloid is connected with considerable difficulties. The case $p = 1$ was considered in the papers ([1]) ($q = 3$), ([3])*, ([4]). In the present note the problem is solved of decomposing the unitary representation of the group G in $L^2(X)$ into irreducible components for odd $p > 1$ and arbitrary $q > 1$, with the aid of spherical (zonal) functions.

2. **Representations of the pseudo-orthogonal group connected with the cone.** Let X_0 be the cone $[x,x] = 0,\ x \neq 0$. The group G acts transitively on the cone.

Let χ denote the pair (σ, ε), where σ is a complex number and ε takes the values 0 and 1. Let D_χ be the space of infinitely differentiable functions $\varphi(\xi)$ on the cone X_0, homogeneous of degree σ and parity ε,

$$\varphi(t\xi) = |t|^\sigma \operatorname{sign}^\varepsilon t \cdot \varphi(\xi),$$

with the usual topology. In D_χ acts the representation T_χ of the group G:

$$T_\chi(g)\varphi(\xi) = \varphi(\xi g).$$

Let $\sigma^* = 2 - n - \sigma,\ \chi^* = (\sigma^*, \varepsilon),\ \bar{\chi}^* = (\bar{\sigma}^*, \varepsilon)$. If σ is not an integer, then T_χ is irreducible and equivalent to T_{χ^*}.

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Let S be the section of the cone by the Euclidean sphere of radius $\sqrt{2}$ with center at the origin; S is the direct product of two Euclidean spheres of unit radius, $S = S^{p-1} \times S^{q-1}$. The space D_χ may be realized as the space of infinitely differentiable functions $\varphi(s)$ on S of parity ε: $\varphi(-s) = (-1)^\varepsilon \varphi(s)$. The Hermitian form

$$(\varphi, \psi) = \int \varphi(s)\overline{\psi(s)}\, ds \tag{1}$$

(where ds is the Euclidean measure on S) on the pair of spaces $D_\chi, D_{\bar{\chi}^*}$ is invariant with respect to G.

The operator A_χ

$$A_\chi \varphi(s) = \frac{1}{\Gamma((q-1)/2 + \sigma^*)} \int \left\{ [s, s']_-^{\sigma^*} + (-1)^\varepsilon [s, s']_+^{\sigma^*} \right\} \varphi(s')\, ds'$$

* In communication ([3]) the contribution of the integral points of the supplementary series to the decomposition formula was omitted; cf. formula (2) of the present note, which refers to the case $p = 1$. In this case there exist two spherical functions corresponding to one representation of the continuous series.

maps D_χ into D_{χ^*} and commutes with the group G. (The integral converges for $\operatorname{Re}\sigma < 3 - n$; for the remaining σ it is analytically continued as a meromorphic function.)

The representations T_χ are unitary in the following cases:

a) **the continuous series:** $\sigma = (2-n)/2 + i\rho$, $-\infty < \rho < \infty$, $\varepsilon = 0, 1$, scalar product (φ, ψ);

b) **the supplementary series:** $(1-n)/2 < \sigma < (3-n)/2$, $\varepsilon = 0, 1$ for odd n; $-n/2 < \sigma < (4-n)/2$, $\varepsilon \equiv n/2 - p$ for even n (here and below the congruence sign denotes congruence modulo 2); scalar product $(A_\chi \varphi, \psi)$.

For integral σ the representations T_χ are, generally speaking, reducible. To clarify the structure of the representations T_χ for integral σ, we use the restriction of T_χ to the maximal compact subgroup K of the group G. K is isomorphic to the direct product of two orthogonal groups: $K = SO(p) \times SO(q)$, and acts transitively on S (each component on its own sphere). In the realization on S, the representation T_χ assigns to elements of the group K rotations: $T_\chi(k)\varphi(s) = \varphi(sk)$, $k \in K$. The representation of the group K by rotations in the space $C^\infty(S)$ decomposes into a direct sum of representations, each of which is realized in the space of homogeneous polynomials harmonic and of degree l in the variables $s_1, \dots, s_p$ and harmonic and of degree m in the variables $s_{p+1}, \dots, s_n$ (see [2], Ch. IX), and is therefore specified by a pair (l, m) of nonnegative integers. We shall call this pair the weight of the representation.

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The restriction of T_χ to K contains representations with weights from the set Λ^ε: $l+m \equiv \varepsilon$. Consider the following four functions, linear with respect to l, m:

$$\beta_1^\sigma(l,m) = \sigma - l - m, \qquad \beta_3^\sigma(l,m) = \sigma + p - 2 + l - m,$$
$$\beta_2^\sigma(l,m) = \sigma + q - 2 - l + m, \quad \beta_4^\sigma(l,m) = \sigma + n - 4 + l + m.$$

If the line $\beta_i^\sigma(l,m) = 0$ intersects Λ^ε, then the subspace $D_{\chi,i}$ in $C^\infty(S)$, containing the spaces of representations with weights from $\{\beta_i^\sigma(l,m) \geq 0\} \cap \Lambda^\varepsilon$, is invariant with respect to G.

From the representations T_χ with integral σ we extract the following two discrete series of unitary representations.

The **first discrete series** is realized, for integral $\sigma > (2-n)/2$, $\varepsilon \equiv p + \sigma$, in the factor spaces $\widetilde{D}_\sigma = D_\chi / D_{\chi,3}$; the scalar product is induced by the form $(\varphi, \psi)_\sigma = (A_\chi \varphi, \psi)$.

The **second discrete series** is realized, for integral $\sigma > (2-n)/2$, $\varepsilon \equiv \sigma + q$, in the factor spaces $D_\chi / D_{\chi,2}$. The representations of the first (respectively, second) discrete series are irreducible for $q > 2$ (respectively, $p > 2$) and decompose into the direct sum of two irreducible representations for $q = 2$ (respectively, $p = 2$).

The representation T'_χ of the group G, contragredient to T_{χ^*}, is constructed in the space D'_χ of functionals on D_{χ^*} (therefore, in the space of generalized functions on S of parity ε) and is given by the formula

$$(T'_\chi(g)F, \varphi) = (F, T_{\chi^*}(g)\varphi), \qquad \varphi \in D_{\chi^*}, \quad F \in D'_\chi.$$

The space D_χ is embedded in D'_χ with the aid of the Hermitian form (1), and T'_χ is an extension of T_χ. The structure of invariant subspaces is transferred from D_χ to D'_χ. In particular, we denote by $\widetilde{D}'_\chi$ the space $D'_\chi / D'_{\chi,1}$. The operator A_χ maps $\widetilde{D}'_\chi$ into D_{χ^*}.

3. **Spherical functions.** Let G_0 be the stationary subgroup of the point $x^0 = (0, \dots, 0, 1) \in X$. We indicate functions from D'_χ invariant with respect to G_0. For the continuous series,

$$\theta_{\rho,\varepsilon}(s) = (s_n)_+^\sigma + (-1)^\varepsilon (s_n)_-^\sigma, \qquad \sigma = (2-n)/2 + i\rho.$$

For the first discrete series ($\chi = \chi_l = (l, \varepsilon)$, where l is an integer $\geq -[\nu]$, $\nu = (n-3)/2$, $\varepsilon \equiv p + l$)

$$\theta_l(s) = \{(s_n)^\sigma + (-1)^\varepsilon (s_n)^\sigma\} / \Gamma(\sigma+1)\big|_{\sigma=l} \quad \text{for } p \equiv 1,$$

$$\theta_l(s) = \{(s_n)^\sigma + (-1)^\varepsilon (s_n)^{-\sigma}\}_{\sigma=l} \quad \text{for } p \equiv 0.$$

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Denote

$$\theta^x_{\rho,\varepsilon} = T'_\chi(g^{-1})\theta_{\rho,\varepsilon}; \qquad \theta^x_l = T'_\chi(g^{-1})\theta_l,$$

where $x = x^0 = x^0 g$. To each finite infinitely differentiable function $f(x)$ on X we associate its Fourier components of the continuous series:

$$F_{\rho,\varepsilon}(s) = \int \theta^x_{\rho,\varepsilon}(s) f(x)\, dx \in D_\chi$$

and of the first discrete series $\widetilde{G}_l$, where $\widetilde{G}_l$ is an element of the factor space $\widetilde{D}_{\chi_l}$ with representative

$$G_l(s) = \int \theta^x_l(s) f(x)\, dx \in D_{\chi_l}.$$

By the spherical function corresponding to the representation of the continuous series (respectively, of the first discrete series) we mean the generalized function $\Phi_{\rho,\varepsilon}$ (respectively Ψ_l) on the hyperboloid X, whose value on $f(x)$ is equal to $(\theta_{\rho,\varepsilon}, F_{\rho,\varepsilon})$ (respectively $(A_{\chi_l}\theta_l, G_l)$).

For $p \equiv 0$, $l \geq 0$, the function θ_l is a polynomial and belongs to the space $D_{\chi,1}$; therefore the spherical function of the first discrete series defined in the preceding paragraph vanishes, and hence the expansion formula given below is proved under the assumption that p is odd.

The main result of the work is the expansion of the delta-function on the hyperboloid in spherical functions for odd $p > 1$

$$\delta = \sum_{l=-[\nu]}^{\infty} a_l \Psi_l + \int_{-\infty}^{\infty} \omega(\rho) \sum_{\varepsilon} \Phi_{\rho,\varepsilon}\, d\rho, \tag{2}$$

where δ is the delta-function concentrated at x^0,

$$a_l = (2l+n-2)\cdot 2^{l-n-3}\pi^{-3/2(n-2)}\Gamma(n-2+l)\Gamma(l+n/2)/\Gamma((n+p-1)/2+l),$$

$$\omega(\rho) = 2^{-n-1}\pi^{-n}\rho\,\operatorname{sh}\pi\rho\cdot|\Gamma((n-2)/2+i\rho)|^2\,.$$

From (2) follows the inversion formula

$$f(x) = \sum_{l=-[\nu]}^{\infty} a_l\, \overline{(A_{\chi_l}\theta^x_l, G_l)} + \int_{-\infty}^{\infty} \omega(\rho) \sum_{\varepsilon} \overline{(\theta^x_{\rho,\varepsilon}, F_{\rho,\varepsilon})}\, d\rho,$$

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where the integral and the series converge absolutely and uniformly with respect to x from any compact subset of X, and the analogue of the Plancherel formula for odd $p > 1$:

$$\int |f(x)|^2\,dx = \sum_{l=-[\nu]}^{\infty} a_l\,(G_l, G_l)_l + \int_{-\infty}^{\infty} \omega(\rho) \sum_{\varepsilon} (F_{\rho,\varepsilon}, F_{\rho,\varepsilon})\,d\rho.$$

4. Explicit expressions for the spherical functions

The spherical functions for $|x_n| \neq 1$ are analytic functions of x_n of parity ε and are expressed in terms of Legendre functions as follows. Continuous series:

$$\Phi_{\rho,\varepsilon}(x) =$$

$$= \begin{cases} c_\sigma (x_n^2-1)^{-\nu/2} \left\{ (\sin(p/2+\nu+\sigma)\pi + \sin(p/2-\nu-\varepsilon)\pi)\, P^\nu_{\nu+\sigma}(x_n) \right. \\ \qquad \left. -\dfrac{2}{\pi} \sin\dfrac{p\pi}{2} \sin(2\nu+\sigma)\pi \cdot Q^\nu_{\nu+\sigma}(x_n) \right\}, & x_n > 1, \\ c_\sigma (1-x_n^2)^{-\nu/2} \sin\dfrac{p\pi}{2} \left\{ P^\nu_{\nu+\sigma}(-x_n) + (-1)^\varepsilon P^\nu_{\nu+\sigma}(x_n) \right\}, & |x_n| < 1, \end{cases}$$

where

$$c_\sigma = 2^{\nu+1}\pi^\nu \Gamma(\sigma+1)\Gamma(-2\nu-\sigma), \qquad \nu = (n-3)/2.$$

The first discrete series for $p = 1,\ q = 1$:

$$\Psi_l(x) = \begin{cases} 0, & |x_n| > 1, \\ b_l\, \dfrac{\pi}{2}\, (-1)^{(q+1)/2} (1-x_n^2)^{-\nu/2} P^\nu_{l+\nu}(x_n), & |x_n| < 1; \end{cases}$$

for $p = 1,\ q = 0$:

$$\Psi_l(x) = \begin{cases} b_l (-1)^{(q-2)/2} (x_n^2-1)^{-\nu/2} e^{-i\pi\nu} Q^\nu_{l+\nu}(x_n), & x_n > 1, \\ b_l (-1)^{(q-2)/2} (1-x_n^2)^{-\nu/2} Q^\nu_{l+\nu}(x_n), & |x_n| < 1, \end{cases}$$

where

$$b_l = 2^{-\nu-1}\pi^{-\nu-2}(2l+n-2)/a_l.$$

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