ON THE CONSTRUCTION OF CUBATURE FORMULAS WITH THE SMALLEST NUMBER OF NODES

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Abstract

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ON THE CONSTRUCTION OF CUBATURE FORMULAS WITH THE SMALLEST NUMBER OF NODES

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Let Ω be a domain in n-dimensional Euclidean space and let $p(x)$ be a weight function, nonnegative for $x \in \Omega$, such that the integrals

$$\mu_{\alpha_1 \cdots \alpha_n} = \int_\Omega p(x) x_1^{\alpha_1} \cdots x_n^{\alpha_n}\, dx$$

exist, the moments of the domain Ω and of the weight function $p(x)$, $\mu_{0 \ldots 0} > 0$.

Consider the problem of constructing a cubature formula

$$\int_\Omega p(x) f(x)\, dx \cong \sum_{j=1}^{N} C_j f(x^{(j)}), \qquad (1)$$

exact for all polynomials of degree not higher than m, where $m = 2l$ is even. The smallest possible number of nodes for which formula (1) can exist is equal to

$$M(n,l) = (n+l)!/n!l!.$$

In what follows we assume that in (1) $N = M(n,l)$.

Renumber the monomials

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \qquad \alpha_1 \geq 0, \ldots, \alpha_n \geq 0,$$

and introduce for them the notation $\{\varphi_i(x)\}_{i=1}^{\infty}$. The numbering is carried out so that monomials of lower degree receive a smaller number, while monomials of the same degree are numbered in any order. In particular, $\varphi_1(x) = 1$.

Write that formula (1) is exact for all polynomials of degree $\leq m$, in other words, exact when $f = \varphi_i(x)$,

$$\sum_{j=1}^{N} C_j \varphi_i(x^{(j)}) = \int_\Omega p(x)\varphi_i(x)\,dx, \qquad i = 1, 2, \dots, M(m, n). \tag{2}$$

We have obtained a system of $M(m, n)$ equations in $(n + 1)M(n, l)$ unknowns.

Introduce three square matrices of order N: the matrix X, determined by the nodes of the cubature formula (1),

$$X = \begin{pmatrix} \varphi_1(x^{(1)}), & \varphi_2(x^{(1)}), \dots, \varphi_N(x^{(1)}) \\ \varphi_1(x^{(2)}), & \varphi_2(x^{(2)}), \dots, \varphi_N(x^{(2)}) \\ . & . & . \\ \varphi_1(x^{(N)}), & \varphi_2(x^{(N)}), \dots, \varphi_N(x^{(N)}) \end{pmatrix};$$

the diagonal matrix C, determined by the coefficients of formula (1),

$$C = [C_1, \dots, C_N]$$

and the Gram matrix of the system $\{\varphi_i(x)\}_{i=1}^N$

$$G = ((\varphi_i, \varphi_k))_{i,k=1}^N. \tag{3}$$

Here it is denoted

$$(\varphi_i, \varphi_k) = \int_\Omega p(x)\varphi_i(x)\varphi_k(x)\,dx.$$

It is known that the system (2) can be written in the form of a single matrix equation [1]

$$X'CX = G. \tag{4}$$

The prime denotes the transposition operation. It follows from equality (4) that the number of nodes N in (1) is the smallest.

The matrix (3) is positive definite, and for it the representation [2]

$$G = \Gamma'\Gamma, \tag{5}$$

holds, where Γ is an upper triangular matrix. The nonuniqueness of the representation (5) reduces to the fact that one may change the sign of all elements of any row of the matrix Γ.

Introduce the notation

$$\Gamma^{-1} = \begin{pmatrix} \beta_{11} & \beta_{12} & \dots & \beta_{1N} \\ 0 & \beta_{22} & \dots & \beta_{2N} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \beta_{NN} \end{pmatrix}.$$

In transforming equation (4), the following assertion is used. The polynomials

$$F_i(x) = \beta_{1i}\varphi_1(x) + \beta_{2i}\varphi_2(x) + \dots + \beta_{ii}\varphi_i(x), \qquad i = 1, 2, \dots, N,$$

form an orthonormal system. In view of what was said above about the nonuniqueness of representation (5), the polynomial $F_i(x)$ is determined by the matrix G up to sign.

Let us give the proof. Denote by $\beta^{(i)}$ the vector coinciding with the i-th column of the matrix Γ^{-1}. We have

$$(F_i, F_j) = \left(\sum_{k=1}^{i} \beta_{ki}\varphi_k, \sum_{l=1}^{j} \beta_{lj}\varphi_l\right) = \sum_{k=1}^{i}\sum_{l=1}^{j} \beta_{ki}\beta_{lj}(\varphi_k, \varphi_l) = (G\beta^{(i)}, \beta^{(j)}) = \\ = (\Gamma'\Gamma\beta^{(i)}, \beta^{(j)}) = (\Gamma\beta^{(i)}, \Gamma\beta^{(j)}) = (e^{(i)}, e^{(j)}) = \delta_i^{(j)}.$$

Here $e^{(i)}$ is the i-th column of the identity matrix, and $\delta_i^{(j)}$ is the Kronecker symbol. The assertion is also true in the case when $\varphi_1, \dots, \varphi_N$ are linearly independent elements of a Hilbert space.

Equation (4) can be brought to the form

$$FF' = C^{-1}, \tag{6}$$

where

$$F = X\Gamma^{-1} = \begin{pmatrix} F_1(x^{(1)}), & F_2(x^{(1)}), & \dots, & F_N(x^{(1)}) \\ F_1(x^{(2)}), & F_2(x^{(2)}), & \dots, & F_N(x^{(2)}) \\ \dots & \dots & \dots & \dots \\ F_1(x^{(N)}), & F_2(x^{(N)}), & \dots, & F_N(x^{(N)}) \end{pmatrix}. \tag{7}$$

Let us note that equation (6) could have been obtained by writing down the fact that the cubature formula (1) is exact for $f = F_{iF}k$, $i, k = 1, 2, \dots, N$.

From (6) it is seen that

$$C_i^{-1} = F_1^2(x^{(i)}) + F_2^2(x^{(i)}) + \dots + F_N^2(x^{(i)}), \qquad i = 1, 2, \dots, N. \tag{8}$$

In particular, the coefficients C_i are positive.

The rows of the matrix (7), by virtue of (6), are orthogonal. If a is one of the nodes of formula (1), then its remaining nodes lie on the algebraic hypersurface

$$F_1(a)F_1(x) + F_2(a)F_2(x) + \dots + F_N(a)F_N(x) = 0. \tag{9}$$

This circumstance can be used in constructing formula (1). For brevity, we shall say that the point a determines the hypersurface (9).

Let us indicate examples of the construction of the cubature formula (1). For $m = 2$ we obtain a new algorithm for constructing a cubature formula exact for polynomials of the second degree. From the system of monomials $1, x_1, \dots, x_n$ we construct an orthonormal system $F_i(x)$, $i = 1, \dots, n+1$. We choose the point $x^{(1)}$ so that among the numbers $F_i(x^{(1)})$, $i = 2, 3, \dots, n+1$, there are some different from zero. Denote by L_1 the hyperplane determined by the point $x^{(1)}$. On L_1 we choose a point $x^{(2)}$ so that the hyperplane L_2 determined by $x^{(2)}$ is not parallel to L_1. At the intersection of L_1 and L_2 we take a point $x^{(3)}$ so that the hyperplane L_3 determined by it is not parallel to any of the hyperplanes L_1 and L_2. At the intersection of L_1, L_2, L_3 we choose a point $x^{(4)}$, and so on. As a result we obtain $n+1$ points $x^{(1)}, \dots, x^{(n+1)}$—the nodes of the cubature formula. Another method of constructing formula (1) for $m = 2$ is considered in ([3]).

Now let $n = 2$, $m = 4$. We assume that the domain Ω is symmetric with respect to both coordinate axes and that the weight has the same symmetry property: $p(x, y) = p(-x, y) = p(x, -y)$ for $(x, y) \in \Omega$. By virtue of the assumptions on Ω and $p(x, y)$, the moment $\mu_{ik} = 0$ if at least one of the numbers i and k is odd.

We shall suppose that the origin $(x_1, y_1) = (0, 0)$ is a node of the desired cubature formula. The curve determined by the point $(0, 0)$ has equation

$$\alpha x^2 + \gamma y^2 = \delta, \tag{10}$$

where $\alpha = \mu_{04}\mu_{20} - \mu_{22}\mu_{02}$, $\gamma = \mu_{40}\mu_{02} - \mu_{22}\mu_{20}$, $\delta = \mu_{40}\mu_{04} - \mu_{22}^2$; here $\delta > 0$ and at least one of the numbers α, γ is positive. If, for example, $\alpha > 0$, then as one more node we take $(x_2, y_2) = (\sqrt{\delta/\alpha}, 0)$. The curve determined by this node,

$$\mu_{04}x^2 - \mu_{22}y^2 + \frac{\alpha}{\mu_{20}}\sqrt{\frac{\delta}{\alpha}}\, x = 0$$

intersects the curve (10) in four real points (x_i, y_i), $i = 3, 4, 5, 6$. The points (x_i, y_i), $i = 1, \dots, 6$, may be taken as the nodes of a cubature formula exact for polynomials of the fourth degree.

For $n = 1$ we obtain a quadrature formula exact for polynomials of degree $m = 2l$ and having $N = l + 1$ nodes. The formula depends on the parameter a—

the abscissa of one of the nodes. The remaining nodes are determined from the algebraic equation (9) of degree $N-1$. Let I be the smallest interval containing Ω. If a lies outside I or coincides with one of its endpoints, then the roots of equation (9) are real, distinct, and lie inside I. If a is one of the roots of the orthogonal polynomial $F_{N+1}(x)$, then the roots of (9) coincide with the remaining roots of $F_{N+1}(x)$. In this case formula (8) is well known ([4]).

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[4] G. Szegő, *Orthogonal Polynomials*, Moscow, 1962.

Note: Figure translations are in progress. See original paper for figures.