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Abstract

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OPTIMAL SOLUTION OF THE PROBLEM OF SYNTHESIS OF A LINEAR ANTENNA

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In this paper the following problem is considered. A nonrealizable pattern $h(\theta)$, $0 \leq \theta \leq \pi$, of a linear antenna is given, together with its length $2l$ and constraints on the distribution of the current in the antenna, guaranteeing the absence of undesirable effects characteristic of superdirective antennas. We shall take these constraints in the form

$$\int_{-l}^{l} |j(x)|^2 \, dx \leq M_0, \qquad \int_{-l}^{l} |j'|^2 \, dx \leq M_1, \tag{1}$$

where M_0, M_1 are given constants.

The meaning of conditions (1) is that the amplitude and the rate of variation of the current along the antenna are bounded on the average. The current distribution $j(x)$ produces the pattern

$$F(\theta) = \frac{\sin\theta}{A} \int_{-l}^{l} j(z) \exp(ik_0 z \cos\theta) \, dz, \qquad 0 \leq \theta \leq \pi, \tag{2}$$

where θ is the angle, measured from the z-axis, along which the antenna is situated, to the radius vector going to the observation point; $k_0 = 2\pi/\lambda = \omega/c$; ω is the oscillation frequency; λ is the wavelength; c is the velocity of propagation of the oscillations; A is a normalizing constant. Introduce the variable $k = k_0 \cos\theta$ and the notation $AF(\theta)/\sin\theta = f(k)$. Then formula (2) takes the form

$$f(k) = \int_{-l}^{l} \exp(ikx) j(x) \, dx, \qquad -k_0 \leq k \leq k_0. \tag{3}$$

We shall call the function $f(k)$ (as well as $F(\theta)$) the radiation pattern produced by the current $j(x)$. The function defined by formula (3) is an entire function (of exponential type with exponent l) of the complex variable k. Denote by

$\Omega(M_0, M_1)$ the set of functions $j(x)$ satisfying conditions (1). It is compact in the space $L_2(-l, l)$ of square-integrable functions on the interval $[-l, l]$.

The problem is as follows: it is required to find, on the set $\Omega(M_0, M_1)$, a function $j(x)$ minimizing the functional

$$G(j) = \int_{-\infty}^{\infty} \left| g(k) - \int_{-l}^{l} \exp(ikx) j(x)\, dx \right|^2 dk = \min, \tag{4}$$

where $g(k) \equiv Ah(\theta)/\sin\theta$, $k = k_0 \cos\theta$.

The function $g(k)$ is thus defined on the interval $-k_0 \leq k \leq k_0$; outside this interval, by definition, we put $g(k) \equiv 0$. Condition (4) means that the realizable pattern produced by the current $j(x)$ approximates the given, generally speaking nonrealizable, pattern in the optimal

in such a way, with the current $j(x)$ varying over such a set that the phenomena characteristic of superdirective antennas are excluded.

The choice of the mean-square criterion of closeness of the patterns in formula (4) is due to computational convenience.

Although the literature devoted to the theory of antenna synthesis is very extensive (see, for example, (^{2,3}) and the references in those works), the problem posed, as far as we know, has not previously been discussed. Apparently, the first indications of the desirability of using optimization in the antenna synthesis problem appeared in (^{2}). In the present paper, a solution of the problem posed is given by reducing it to a convex programming problem. Let us proceed to the solution of the problem posed.

Denote by $\tilde{g}(x)$ the Fourier transform of the function $g(k)$:

$$g(k) = \int_{-\infty}^{\infty} e^{ikx} \tilde{g}(x)\, dx. \tag{5}$$

Applying Parseval' s equality to the integral (4) and taking into account that $j(x) = 0$ for $|x| > l$, we obtain

$$G(j) = \int_{-\infty}^{\infty} |\tilde{g}(x) - j(x)|^2\, dx =$$

$$= \int_{|x|>l} |\tilde{g}(x)|^2\, dx + \int_{-l}^{l} |\tilde{g}(x) - j(x)|^2\, dx = \delta(g, l) + I(j) = \min, \tag{6}$$

where $\delta(g, l) > 0$ does not depend on $j(x)$ and, consequently, if the antenna length l and the pattern $g(k)$ are fixed, then $\delta(g, l)$ characterizes the optimal approximation to the unrealizable pattern. The functional $G(j)$ is minimized simultaneously with the functional $I(j)$. In order to reduce the problem of finding $\min I$ on the set of functions $\Omega(M_0, M_1)$ to a convex programming problem, expand the functions $\tilde{g}(x)$ and $j(x)$ on the interval $(-l, l)$ in a Fourier series

$$\tilde{g}(x) = \sum_{n=-\infty}^{\infty} \tilde{g}_n \exp\left(in\frac{\pi}{l}x\right), \qquad j(x) = \sum_{n=-\infty}^{\infty} j_n \exp\left(in\frac{\pi}{l}x\right). \tag{7}$$

Applying Parseval’ s equality for Fourier series to the integral I, we obtain

$$I(j) = \sum_{n=-\infty}^{\infty} |j_n - \tilde{g}_n|^2 = \min. \tag{8}$$

We write the conditions (1) in the form

$$\sum_{n=-\infty}^{\infty} |j_n|^2 \le M_0, \qquad \sum_{n=-\infty}^{\infty} n^2 |j_n|^2 \le M_1 \frac{l^2}{\pi^2} = M_2^*. \tag{9}$$

Problem (8)–(9) is a convex programming problem with objective function (8), depending on a countable set of variables. Since the theory is constructed (4,5) for functions of a finite number of variables, for its application we consider the problem

$$I_N(j) = \sum_{n=-N}^{N} |j_n - \tilde{g}_n|^2 = \min, \tag{10}$$

$$\sum_{n=-N}^{N} |j_n|^2 \le M_0, \qquad \sum_{n=-N}^{N} n^2 |j_n|^2 \le M_2. \tag{11}$$

As $N \to \infty$, problem (10)–(11) reduces to problem (8)–(9). For sufficiently large N, in view of the convergence of the series (8), (9), the solution of problem (10)–(11) differs arbitrarily little from the solution of problem (8)–(9). Finally,

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[*] The second condition (3) is equivalent to the second condition (1) if $j(l) = j(-l)$. Otherwise there is no equivalence, since the Fourier coefficient of the function $j'(x)$ is not equal to $i\pi n j_n / l$. Therefore, if $j(-l) \ne j(l)$, then instead of the system of functions $\{\exp(in\pi x/l)\}$ in (7) one should use, for expansion, the complete orthogonal system of functions $\{\cos(n\pi x/2l + \pi n/2)\}$ on the interval $[-l, l]$. In this case, as is not difficult to verify, the entire scheme for solving the problem is completely preserved.

we note that the solution of problem (8)–(9) exists and is unique. Existence is proved by a standard argument in mathematics: a sequence minimizing the functional $I(j)$ converges, since the set (9) is compact in l_2 [^1]. The functional $I(j)$ is continuous (in the sense of convergence in the mean square); therefore its values on the elements of a minimizing sequence tend to the minimum of this functional, which we denote by the letter d. The limit of the minimizing sequence belongs to the admissible set (9), since it is closed. The uniqueness of the solution of problem (8)–(9) follows from the strict convexity of the functional (8) with respect to the variables j_n, $n = 0, \pm 1, \dots, \pm N$. If there were two distinct solutions $\{j_n\}$ and $\{j'_n\}$, then

$$I\left(\frac{\{j_n\} + \{j'_n\}}{2}\right) < \frac{I(\{j_n\}) + I(\{j'_n\})}{2} = d,$$

which is impossible, since $d \leq I(\{j_n\})$ for any admissible sequence $\{j_n\}$.

The existence and uniqueness of the solution of the finite-dimensional problem (10)–(11) follow, respectively, from the Weierstrass theorem stating that a continuous function (10) attains its minimum on the closed set (11), and from the strict convexity of the function (10).

We apply to problem (10)–(11) the Kuhn–Tucker theorem ([^4], p. 88). As a result we obtain that, for the values of the variables j_n minimizing the functional (10), the inequalities (11) become equalities (here it is assumed that the numbers $\tilde{g}_n$ do not satisfy the constraints (11)). In addition, the equalities

$$\partial\Phi/\partial j_n = 0, \qquad \partial\Phi/\partial j_n^* = 0, \qquad -N \leq n \leq N, \tag{12}$$

must hold, where j_n^* is the quantity complex-conjugate to j_n;

$$\Phi = \sum_{n=-N}^{N} |j_n - \tilde{g}_n|^2 + u_0 \left(\sum_{n=-N}^{N} |j_n|^2 - M_0 \right) + u_1 \left(\sum_{n=-N}^{N} n^2 |j_n|^2 - M_2 \right), \tag{13}$$

and u_0, u_1 are real constants–the Lagrange multipliers.

Thus, from the two conditions (11), in which the inequality sign must be replaced by the equality sign, and from the $2(2N+1)$ conditions (12), one can find the $2N+1$ complex variables j_n and the Lagrange multipliers u_0, u_1. If the numbers $\tilde{g}_n$, $-N \leq n \leq N$, satisfy conditions (11), then problem (10)–(11) is solved trivially: $j_n = \tilde{g}_n$. We note that in the book [^5], p. 284, an algorithm is given for solving the general problem of convex programming that does not use the analytical Kuhn–Tucker conditions. This algorithm has been tested on computers. A description of the algorithm mentioned requires much space and therefore is not given here.

Equations (12) in the case under consideration are easily solvable, since they lead to a linear system with a diagonal matrix.

The solution of this system has the form

$$j_n = \frac{\tilde{g}_n}{1 + u_0 + u_1 n^2}, \qquad -N \leq n \leq N. \tag{14}$$

The constants u_0 and u_1 should be determined from conditions (11):

$$\sum_{n=-N}^{N} |\tilde{g}_n|^2 \frac{1}{(1 + u_0 + u_1 n^2)^2} = M_0, \tag{15}$$

$$\sum_{n=-N}^{N} |\tilde{g}_n|^2 \frac{n^2}{(1 + u_0 + u_1 n^2)^2} = M_1. \tag{16}$$

Equations (15)—(16) should be solved by numerical methods.

Let us note in conclusion that the proposed method for solving the problem of synthesizing a linear antenna is stable with respect to small changes of the pattern, since the set on which the minimizing function is sought is compact ([5]).

It is useful to point out that equations (14)—(16) can be obtained by the Ritz method in solving the problem of minimizing the functional $I(j)$ under conditions (1). It can be proved that the function solving the indicated problem turns the inequality signs in (1) into equality signs. We apply the Ritz method to the resulting isoperimetric problem. As coordinate functions it is convenient to use the system of functions $\{\exp(in\frac{\pi}{l}x)\}$, if $j(l) = j(-l)$, and $\{\cos(\frac{n\pi}{2l}x + \frac{n\pi}{2})\}$, if $j(l) \neq j(-l)$.

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