ON NEIGHBORING COEFFICIENTS OF UNIVALENT FUNCTIONS

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Abstract

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MATHEMATICS

I. M. MILIN

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Let S be the class of functions

$$f(\zeta) = \zeta + \sum_{n=2}^{\infty} C_n \zeta^n,$$

regular and univalent in the disk $|\zeta| < 1$, and let Σ be the class of functions

$$F(z) = z + a_0 + a_1 z^{-1} + \cdots,$$

meromorphic and univalent in the domain $|z| > 1$. Each function $F(z) \in \Sigma$ generates a system of functions $\{A_n(z)\}$ $(n = 1, 2, \ldots)$ by means of the expansion

$$\ln \frac{z - t}{F(z) - F(t)} = \sum_{n=1}^{\infty} A_n(t) z^{-n}, \qquad |z, t| > 1. \tag{1}$$

Among the properties of the systems $\{A_n(z)\}$ (1), we recall the inequality

$$\sum_{n=1}^{\infty} n |A_n(z)|^2 \leq \ln \frac{1}{1 - r^2}, \qquad |z| = \frac{1}{r} > 1. \tag{2}$$

For the coefficients of functions of the class S, G. M. Goluzin (2) obtained the inequality

$$\big| |c_{n+1}| - |c_n| \big| < A n^{1/4} \ln n, \qquad n \geq 2,$$

where A is an absolute constant. This estimate was improved by M. Bernatskii, and then Hayman achieved the result (3)

$$\big| |c_{n+1}| - |c_n| \big| < A \qquad (n = 1, 2, \ldots). \tag{3}$$

For odd functions

$$f(\zeta) = \sum_{n=0}^{\infty} c_{n+1} \zeta^{2n+1} \in S,$$

K. Lucas (4) proved that

$$\big||c_{n+1}| - |c_n|\big| = O(n^{\beta}), \qquad \beta = -(\sqrt{2} - 1) \tag{4}$$

(according to Hayman' s conjecture (3), $\beta = -1/2 + \varepsilon$).

Below we give a simple derivation of inequality (3) with an estimate of A, based on property (2) of the systems $\{A_n(z)\}$, and under a certain condition we prove relation (4) for every odd function $f(\zeta) \in S$ with $\beta = -1/2$.

Lemma (5,6). Let $\{A_k\}_1^{\infty}$ be an arbitrary sequence of complex numbers, generating the sequence $\{D_k\}_0^{\infty}$ by the formal expansion

$$\exp\left\{\sum_{k=0}^{\infty} A_k z^k\right\} = \sum_{k=0}^{\infty} D_k z^k. \tag{5}$$

Then $(n = 1, 2, \ldots)$

$$|D_n| \le \exp\left\{\frac{1}{2}\left(\sum_{k=1}^{n} k|A_k|^2 - \sum_{k=1}^{n} \frac{1}{k}\right)\right\}, \tag{6}$$

$$\sum_{k=0}^{n} |D_k|^2 \le (n+1)\exp\left\{\sum_{k=1}^{n+1} k|A_k|^2 - \sum_{k=1}^{n+1} \frac{1}{k} + 1 - \frac{1}{n+1}\sum_{k=1}^{n+1} k^2|A_k|^2\right\}. \tag{7}$$

$$\sum_{k=0}^{\infty} |D_k|^2 \le \exp\left\{\sum_{k=1}^{\infty} k|A_k|^2\right\}. \tag{8}$$

The equality sign in (6) and (7) occurs if and only if

$$A_k = \frac{1}{k}\eta^k \quad (k = 1, 2, \ldots, n), \quad |\eta| = 1,$$

and in (8)–if and only if

$$A_k = \frac{1}{k}c^k \quad (k = 1, 2, \ldots), \quad |c| < 1.$$

Theorem 1. For functions $f(\zeta) = \sum_{n=1}^{\infty} c_n \zeta^n \in S$, for every n $(n = 1, 2, \ldots)$ the inequality (3) holds, where $A < 9$.

Proof. From the function $f(\zeta) \in S$ construct the function $F(z) \in \Sigma$:

$$F(z) = \frac{1}{f(1/z)}, \quad |z| > 1. \tag{9}$$

Put

$$\rho = 1/r = 1 + 1/n, \tag{10}$$

choose some value t on the circle $|z| = \rho$ from the condition

$$|F(t)| = \min_{|z|=\rho} |F(z)| \tag{11}$$

and form the function $(1 - t/z)f'(1/z)$.

Cauchy’ s formula for the Taylor coefficients of this function gives

$$(n+1)c_{n+1} - nc_{nt} = \frac{1}{2\pi i} \int_{|z|=\rho} \left(1 - \frac{t}{z}\right) f'\left(\frac{1}{z}\right) z^n \frac{dz}{z}$$

$$= \frac{1}{2\pi i} \int_{|z|=\rho} \left(1 - \frac{F(t)}{F(z)}\right)^2 \frac{zF''(z)}{F(z) - F(t)} \frac{z - t}{F(z) - F(t)} z^n \frac{dz}{z}. \tag{12}$$

Under the integral sign in (12), the functions

$$\frac{zF''(z)}{F(z) - F(t)} \quad \text{and} \quad \frac{z - t}{F(z) - F(t)}$$

may be replaced by the n-th partial sums of their Taylor expansions about $z = \infty$. From (1) we find the expansion of the first function, namely:

$$\frac{zF''(z)}{F(z) - F(t)} = 1 + \sum_{k=1}^{\infty} (t^k + kA_k(t)) z^{-k} = \sum_{k=0}^{\infty} (t^k + kA_k(t)) z^{-k},$$

and for the expansion of the second we introduce the notation

$$\frac{z - t}{F(z) - F(t)} = \exp\left\{\sum_{k=1}^{\infty} A_k(t) z^{-k}\right\} = \sum_{k=0}^{\infty} D_k(t) z^{-k}.$$

Then (12) is rewritten as follows:

$$(n+1)c_{n+1} - nc_{nt} = \frac{1}{2\pi i} \int_{|z|=\rho} \left(1 - \frac{F(t)}{F(z)}\right)^2 \sum_{k=0}^{n} (t^k + kA_k(t)) z^{-k}$$

$$\times \sum_{k=0}^{n} D_k(t) z^{-k} z^n \frac{dz}{z}. \tag{13}$$

From (13) and (10) we obtain the initial inequality

$$||c_{n+1}| - |c_n|| \leq \frac{1}{n+1} \frac{\rho^n}{2\pi} \int_{|z|=\rho} \left| \left(1 - \frac{F(t)}{F(z)}\right)^2 \sum_{k=0}^{n} (t^k + kA_k(t)) z^{-k} \right.$$

$$\left. \times \sum_{k=0}^{n} D_k(t) z^{-k} \frac{dz}{z} \right|. \tag{14}$$

But on the circle $|z| = \rho$, by virtue of (11), $|1 - F(t)/F(z)| \leq 2$, and therefore from (14), with the aid of Bunyakovsky' s inequality, we shall have

$$||c_{n+1}| - |c_n|| \leq \frac{4\rho^n}{n+1} \left(\sum_{k=0}^{n} |t^k + kA_k(t)|^2 r^{2k} \sum_{k=0}^{n} |D_k(t)|^2 r^{2k} \right)^{1/2}. \tag{15}$$

Further, obviously,

$$|t^k + kA_k(t)|^2 \leq 2\rho^{2k} + 2k^2 |A_k(t)|^2,$$

therefore (15) may be written in the form

$$||c_{n+1}| - |c_n|| \leq \frac{4\sqrt{2}\rho^n}{n+1} \left[(n+1) \sum_{k=0}^{n} |D_k(t)|^2 + r^2 \sum_{k=1}^{n+1} k^2 |A_k(t)|^2 \sum_{k=0}^{n} |D_k(t)|^2 \right]^{1/2}. \tag{16}$$

We estimate the first term in the square brackets in (16) with the aid of inequality (8) of the lemma, and to estimate the second we apply (9) and the obvious inequality $xe^{1-x} \leq 1$, where $x = \frac{1}{n+1} \sum_{k=1}^{n+1} k^2 |A_k(t)|^2$. Then we obtain

$$||c_{n+1}| - |c_n|| \leq \frac{4\sqrt{2}\rho^n}{n+1} \left[(n+1) \exp\left\{ \sum_{k=1}^{\infty} k|A_k(t)|^2 \right\} + r^2 (n+1)^2 \exp\left\{ \sum_{k=1}^{n+1} k|A_k(t)|^2 - \sum_{k=1}^{n+1} \frac{1}{k} \right\} \right]^{1/2}.$$

According to (2),

$$\sum_{k=1}^{\infty} k|A_k(t)|^2 \leq \ln \frac{1}{1-r^2}, \qquad \sum_{k=1}^{n+1} \frac{1}{k} > \ln(n+1) + c$$

(c is Euler' s constant), and therefore we have

$$||c_{n+1}| - |c_n|| \leq \frac{4\sqrt{2}\rho^n}{n+1} (1 + r^2 e^{-c})^{1/2} \left(\frac{n+1}{1-r^2} \right)^{1/2} < 4e(1 + e^{-c})^{1/2}. \tag{17}$$

Inequality (17) gives for the constant A the estimate $A < 14$. The desired estimate is obtained in an analogous manner if one starts from the identity

$$\left(1 - \frac{t}{z}\right) f'\left(\frac{1}{z}\right) = \frac{zF'(z)}{F(z)} \frac{z-t}{F(z)-F(t)} - \frac{1}{z} f'\left(\frac{1}{z}\right) \frac{z-t}{F(z)-F(t)} F(t)$$

and can easily be improved by additional computations.

Further, for

$$f(\zeta) = \sum_{n=0}^{\infty} c_{n+1}\zeta^{2n+1} \in S$$

we denote

$$M(r,f) = \max_{|\zeta|=r} |f(\zeta)|, \qquad 0 < r < 1,$$

$$\alpha = \lim_{r\to 1}(1-r^2)M(r,f), \qquad \ln\frac{f(\sqrt{\xi})}{\sqrt{\xi}} = \sum_{k=1}^{\infty}\gamma_k\xi^k. \tag{18}$$

Theorem 2. For every odd function

$$f(\xi) = \sum_{n=0}^{\infty} c_{n+1}\xi^{2n+1} \in S,$$

for which $\alpha \neq 0$, we have

$$\big||c_{n+1}| - |c_n|\big| < \alpha^{-1/4}n^{-1/2}. \tag{19}$$

Proof. I. E. Bazilevich [7] proved the inequality

$$\sum_{k=1}^{\infty} k\left|\gamma_k - \frac{1}{k}t^k\right|^2 \leq \frac{1}{2}\ln\frac{1}{\alpha}, \tag{20}$$

where t is some number, $|t| = 1$. Taking (18) into account, one can write the expansions

$$\ln\left[\frac{f(\sqrt{\zeta})}{\sqrt{\zeta}}(1-t\zeta)\right] = \sum_{k=1}^{\infty}\left(\gamma_k - \frac{1}{k}t^k\right)\zeta^k,$$

$$\frac{f(\sqrt{\zeta})}{\sqrt{\zeta}}(1-t\zeta) = \sum_{k=0}^{\infty}(c_{k+1} - tc_k)\zeta^k, \qquad c_0 = 0.$$

Now, for the sequence

$$\left\{A_k = \gamma_k - \frac{1}{k}t^k\right\} \qquad (k = 1, 2, \ldots)$$

we apply inequality (6) of the lemma:

$$|c_{n+1} - tc_n| \leq \exp\left\{\frac{1}{2}\left(\sum_{k=1}^{n} k\left|\gamma_k - \frac{1}{k}t^k\right|^2 - \sum_{k=1}^{n}\frac{1}{k}\right)\right\},$$

which, together with (20), gives (19). The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.

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