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Abstract

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ON THE THEORY OF BIORTHOGONAL EXPANSIONS OF FIELDS OF METEOROLOGICAL ELEMENTS

In recent years, questions concerning the representation of fields of meteorological elements in terms of the eigenfunctions of the correlation matrix of hydrodynamic elements, and the use of these functions for weather-forecasting purposes, have been widely discussed in the literature. The basis of such an approach was laid in works ($^{1-3}$). Alongside this, attempts have been made to construct a system of eigenfunctions directly on the basis of an analysis of the structure of the operators of the problems of dynamical meteorology ($^{4-7}$). This approach led to a biorthogonal system of functions that describes well the vertical structure of meteorological fields. In the present work, on the basis of an analysis of the structure of the equations of the dynamics of atmospheric processes, principles are developed for constructing a biorthogonal system of eigenfunctions in three-dimensional Euclidean space, and questions of their use for weather-forecasting purposes are discussed. For simplicity, let us assume that the domain in which the solution of the hydrodynamic equations is defined coincides with a rectangular parallelepiped D in the coordinate system x, y, z, where the plane $z = 0$ coincides with sea level, and $z = h$ with the level of the tropopause. Suppose that at sea level and at the level of the tropopause the vertical currents are equal to zero. With a view to generalizing the results to the sphere, let us assume that on the remaining faces of the parallelepiped the periodicity conditions for the solution are satisfied. With respect to the Coriolis parameter, let us assume that it is a linear function of y. We shall consider that the daytime surface of the Earth is covered by continents and oceans, and that the basic state of the atmosphere is the climate, against the background of which essentially nonstationary processes develop, determined by the initial conditions of the problem. The main task consists in finding a complete system of eigenfunctions for atmospheric processes of planetary scale, the elements of which are carriers of a large amount of meaningful information about long-acting factors, such as the inhomogeneity in the heat exchange of the Earth' s surface due to continents and oceans, the climatic field of wind, temperature, and pressure, and large-scale turbulence. For this purpose we shall use the linearized system of equations of hydrothermodynamics of atmospheric processes about the climatic state. Information about the climate is assumed to be known. The climatic values of the hydrodynamic elements will be regarded as quasi-stationary for the period under study and will be denoted by an overbar. Then the system of hydrodynamic equations may be represented in the form

$$
\begin{gathered}
\frac{\partial u}{\partial t} + Ku - lv = -\frac{1}{\bar\rho}\frac{\partial p}{\partial x} + \Pi u - (\operatorname{grad}\bar u, \mathbf{u}),\\
\frac{\partial v}{\partial t} + Kv + lu = -\frac{1}{\bar\rho}\frac{\partial p}{\partial y} + \Pi v - (\operatorname{grad}\bar v, \mathbf{u}),\\
\frac{\partial w}{\partial t} + Kw + g\frac{\rho}{\bar\rho} = -\frac{1}{\bar\rho}\frac{\partial p}{\partial z} + \Pi w - (\operatorname{grad}\bar w, \mathbf{u}),\\
\frac{\partial \rho}{\partial t} + \operatorname{div}\bar\rho\mathbf{u} = 0,\\
\frac{\partial \vartheta}{\partial t} + K\vartheta + \gamma_a w = \Pi\vartheta - (\operatorname{grad}\bar\vartheta, \mathbf{u}),\\
\frac{p}{\bar p} = \frac{T}{\bar T} + \frac{\rho}{\bar\rho}, \qquad \vartheta = T - \frac{\gamma_a}{g\bar\rho}\,p,
\end{gathered}
\tag{1}
$$

where u, v, w, p, ρ, and T are deviations of the corresponding quantities from their climatic value, and the operators K and Π are defined by the relations

$$
K\varphi = \bar u\frac{\partial\varphi}{\partial x} + \bar v\frac{\partial\varphi}{\partial y} + \bar w\frac{\partial\varphi}{\partial z}, \qquad \Pi\varphi = \frac{\partial}{\partial z}\nu\frac{\partial\varphi}{\partial z} + \mu\Delta\varphi.
$$

To the system of equations (1) we adjoin homogeneous boundary conditions

$$
\begin{gathered}
u = 0, \qquad v = 0, \qquad w = 0, \qquad \nu\,\partial w/\partial z = 0, \qquad \nu\,\partial\vartheta/\partial z = \alpha\vartheta \quad \text{for } z = 0;\\
\nu\,\partial u/\partial z = 0, \qquad \nu\,\partial v/\partial z = 0, \qquad w = 0, \qquad \nu\,\partial w/\partial z = 0, \qquad \nu\,\partial\vartheta/\partial z = 0 \quad \text{for } z = h,
\end{gathered}
\tag{2}
$$

where α is a prescribed heat-transfer coefficient.

From the last two equations of system (1) we find

$$
\frac{p}{\bar p} = \frac{1}{1-\chi}\left(\frac{\rho}{\bar\rho} + \frac{\vartheta}{\bar T}\right), \qquad \frac{T}{\bar T} = \frac{1}{1-\chi}\left(\chi\frac{\rho}{\bar\rho} + \frac{\vartheta}{\bar T}\right), \qquad \chi = \frac{\gamma_a R}{g}. \tag{3}
$$

We next eliminate the functions $p/\overline{p}$ and $T/\overline{T}$ from the system of equations (1):

$$\frac{\partial u}{\partial t} + Ku - lv = -\frac{\sigma}{\overline{\rho}}\frac{\partial}{\partial x}(\overline{T}\rho + \overline{\rho}\vartheta) + \Pi u - (\operatorname{grad}\overline{u}, \mathbf{u}),$$

$$\frac{\partial v}{\partial t} + Kv + lu = -\frac{\sigma}{\overline{\rho}}\frac{\partial}{\partial y}(\overline{T}\rho + \overline{\rho}\vartheta) + \Pi v - (\operatorname{grad}\overline{v}, \mathbf{u}),$$

$$\frac{\partial w}{\partial t} + Kw + g\frac{\rho}{\overline{\rho}} = -\frac{\sigma}{\overline{\rho}}\frac{\partial}{\partial z}(\overline{T}\rho + \overline{\rho}\vartheta) + \Pi w - (\operatorname{grad}\overline{w}, \mathbf{u}), \tag{4}$$

$$\partial\rho/\partial t + \partial\overline{\rho}u/\partial x + \partial\overline{\rho}v/\partial y + \partial\overline{\rho}w/\partial z = 0,$$

$$\partial\vartheta/\partial t + K\vartheta + (\gamma_a - \overline{\gamma})w = \Pi\vartheta - (\operatorname{grad}\overline{\vartheta}, \mathbf{u}),$$

where $\sigma = R/(1-\chi)$.

We introduce for consideration a vector-function $\vec{\varphi}$ with components u, v, w, ρ, ϑ and the operator

$$A = \left\| \begin{array}{ccccc} K + \dfrac{\partial\overline{u}}{\partial x} - \Pi & \dfrac{\partial\overline{u}}{\partial y} - l & \dfrac{\partial\overline{u}}{\partial z} & \dfrac{\sigma}{\overline{\rho}}\dfrac{\partial}{\partial x}(\overline{T}\cdot) & \dfrac{\sigma}{\overline{\rho}}\dfrac{\partial}{\partial x}(\overline{\rho}\cdot) \\ \dfrac{\partial\overline{v}}{\partial x} + l & K + \dfrac{\partial\overline{v}}{\partial y} - \Pi & \dfrac{\partial\overline{v}}{\partial z} & \dfrac{\sigma}{\overline{\rho}}\dfrac{\partial}{\partial y}(\overline{T}\cdot) & \dfrac{\sigma}{\overline{\rho}}\dfrac{\partial}{\partial y}(\overline{\rho}\cdot) \\ \dfrac{\partial\overline{w}}{\partial x} & \dfrac{\partial\overline{w}}{\partial y} & K + \dfrac{\partial\overline{w}}{\partial z} - \Pi & \dfrac{\sigma}{\overline{\rho}}\dfrac{\partial}{\partial z}(\overline{T}\cdot) + \dfrac{g}{\overline{\rho}} & \dfrac{\sigma}{\overline{\rho}}\dfrac{\partial}{\partial z}(\overline{\rho}\cdot) \\ \dfrac{\partial}{\partial x}(\overline{\rho}\cdot) & \dfrac{\partial}{\partial y}(\overline{\rho}\cdot) & \dfrac{\partial}{\partial z}(\overline{\rho}\cdot) & 0 & 0 \\ \dfrac{\partial\overline{\vartheta}}{\partial x} & \dfrac{\partial\overline{\vartheta}}{\partial y} & \dfrac{\partial\overline{\vartheta}}{\partial z} + \gamma_a & 0 & K - \Pi \end{array} \right\|.$$

Let us note that here $\partial\overline{u}/\partial x_i$, $\partial\overline{v}/\partial x_i$, and $\partial\overline{w}/\partial x_i$ are prescribed functions, and the dot in parentheses means that the differential operators act on a product of functions. Suppose that the vector-function $\vec{\varphi}$ belongs to a Hilbert space R, whose elements satisfy the boundary conditions (2) and the requirement of periodicity on the other faces of the parallelepiped.

We shall denote the components of the vector-function $\vec{\varphi}$ by φ_j $(j = 1, 2, \dots, 5)$, and the adjoint functions by $\vec{\varphi}^*$, whose elements, by their components, have functions u^*, v^*, w^*, ρ^* and ϑ^*, satisfying the following conditions:

$$u^* = 0, \quad v^* = 0, \quad w^* = 0, \quad \nu\,\partial w^*/\partial z = 0, \quad \nu\,\partial \vartheta^*/\partial z = a\vartheta^* \quad \text{for } z = 0; \tag{5}$$

$$\nu\,\partial u^*/\partial z = 0, \quad \nu\,\partial v^*/\partial z = 0, \quad \nu\,\partial w^*/\partial z = 0, \quad w^* = 0, \quad \nu\,\partial \vartheta^*/\partial z = 0 \quad \text{for } z = h.$$

We shall assume the vector functions $\vec{\varphi}^*$ to be complex; the components of the vector function $\vec{\varphi}^*$ will be denoted by φ_j^* $(j = 1, 2, \dots, 5)$. Define the scalar product of the functions $\vec{\varphi}$ and $\vec{\varphi}^*$ in the form

$$(\vec{\varphi}, \vec{\varphi}^*) = \sum_j \iiint_D \varphi_j \overline{\varphi_j^*}\, dD, \tag{6}$$

where the bar above denotes the complex conjugate quantity. Along with the operator A, introduce into consideration the adjoint operator A^*, defined by Lagrange’ s identity

$$(\vec{\varphi}^*, A\vec{\varphi}) = (A^*\vec{\varphi}^*, \vec{\varphi}), \tag{7}$$

where $\vec{\varphi} \in R$, and $\vec{\varphi}^* \in R^*$. As a result of transformations we arrive at the operator A^*

$$A^* = \left\| \begin{matrix} -K^* + \dfrac{\partial \bar{u}}{\partial x} - \Pi & \dfrac{\partial \bar{v}}{\partial x} + l & \dfrac{\partial \bar{w}}{\partial x} & -\bar{\rho}\dfrac{\partial}{\partial x} & \dfrac{\partial \bar{\vartheta}}{\partial x} \\ \dfrac{\partial \bar{u}}{\partial y} - l & -K^* + \dfrac{\partial \bar{v}}{\partial y} - \Pi & \dfrac{\partial \bar{w}}{\partial y} & -\bar{\rho}\dfrac{\partial}{\partial y} & \dfrac{\partial \bar{\vartheta}}{\partial y} \\ \dfrac{\partial \bar{u}}{\partial z} & \dfrac{\partial \bar{v}}{\partial z} & -K^* + \dfrac{\partial \bar{w}}{\partial z} - \Pi & -\bar{\rho}\dfrac{\partial}{\partial z} & \dfrac{\partial \bar{\vartheta}}{\partial z} + \chi_a \\ -\sigma\bar{T}\dfrac{\partial}{\partial x}\left(\dfrac{1}{\rho}\cdot\right) & -\sigma\bar{T}\dfrac{\partial}{\partial y}\left(\dfrac{1}{\rho}\cdot\right) & -K^* + \dfrac{\partial}{\partial z}\left(\dfrac{1}{\rho}\cdot\right) + \dfrac{g}{\rho} & 0 & 0 \\ -\sigma\bar{\rho}\dfrac{\partial}{\partial x}\left(\dfrac{1}{\rho}\cdot\right) & -\sigma\bar{\rho}\dfrac{\partial}{\partial y}\left(\dfrac{1}{\rho}\cdot\right) & \sigma\bar{\rho}\dfrac{\partial}{\partial z}\left(\dfrac{1}{\rho}\cdot\right) & 0 & -K^* - \Pi \end{matrix} \right\|,$$

where the notation for the operator K^* has been introduced:

$$K^* = \frac{\partial}{\partial x}(\bar{u}\cdot) + \frac{\partial}{\partial y}(\bar{v}\cdot) + \frac{\partial}{\partial z}(\bar{w}\cdot).$$

We now consider the spectral problems for the operators A and A^*

$$A\vec{\psi} = \lambda\vec{\psi}, \qquad A^*\vec{\psi}^* = \bar{\lambda}\,\vec{\psi}^*, \tag{8}$$

where $\bar{\lambda}$ is the complex conjugate number with respect to λ.

Suppose that the homogeneous problems (8) determine two complete systems of eigenfunctions $\{\vec{\psi}_n\}$ and $\{\vec{\psi}^*_m\}$, and the corresponding eigenvalues $\{\lambda_n\}$, $\{\bar{\lambda}_m\}$. It is not difficult to verify that the eigenfunctions $\vec{\psi}_n$ and $\vec{\psi}^*_m$ are biorthogonal. If these functions are normalized in a suitable way, then we shall have

$$(\vec{\psi}_n, \vec{\psi}^*_m) = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases} \tag{9}$$

By virtue of the assumed completeness of the systems of functions, it is possible to represent any function $\vec{\varphi} \in R$ in the form of the series

$$\vec{\varphi} = \sum_n \varphi_n \vec{\psi}_n, \tag{10}$$

where φ_n are Fourier coefficients determined by the relation

$$\varphi_n = (\vec{\varphi}, \vec{\psi}^*_n). \tag{11}$$

Having a set of eigenfunctions, we can approach the solution of the weather-forecasting problem on the basis of the dynamical-statistical method. In fact, expanding the fields of meteorological elements in the system of eigenfunctions $\vec{\psi}_n$ at various moments of time preceding the forecast lead time, we arrive at the problem of temporal extrapolation of random processes. In this case the random numbers of the time series are the Fourier coefficients themselves in the expansion of the vector function $\vec{\varphi}$ in a series in $\vec{\psi}_n$.

One may hope that, possessing great information capacity, even the first few eigenfunctions $\vec{\psi}_n$ and $\vec{\psi}^*_m$ will make it possible to describe the evolution of large-scale processes sufficiently well. Thus, the basic problem in this direction of research is the solution of the spectral problems (8), which in component form have the form

$$
\begin{aligned}
Ku + (\operatorname{grad}\bar{u}, \mathbf{u}) - \Pi u - lv + \frac{\sigma}{\rho}\frac{\partial \bar{T} p}{\partial x} + \frac{\sigma}{\rho}\frac{\partial p\vartheta}{\partial x} &= \lambda u,\\
Kv + (\operatorname{grad}\bar{v}, \mathbf{u}) - \Pi v + lu + \frac{\sigma}{\rho}\frac{\partial \bar{T} p}{\partial y} + \frac{\sigma}{\rho}\frac{\partial p\vartheta}{\partial y} &= \lambda v,\\
Kw + (\operatorname{grad}\bar{w}, \mathbf{u}) - \Pi w + \frac{g}{\rho}\rho + \frac{\sigma}{\rho}\frac{\partial \bar{T} p}{\partial z} + \frac{\sigma}{\rho}\frac{\partial p\vartheta}{\partial z} &= \lambda w,\\
\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} &= \lambda \rho,\\
K\vartheta + (\operatorname{grad}\bar{\vartheta}, \mathbf{u}) - \Pi\vartheta + \gamma_a w &= \lambda\vartheta
\end{aligned}
\tag{12}
$$

in the case of the system of basic equations, and

$$
\begin{aligned}
-K^* u^* + \left(\frac{\partial \bar{\mathbf{u}}}{\partial x}, \mathbf{u}^*\right) - \Pi u^* + lv^* - \bar{\rho}\frac{\partial p^*}{\partial x} + \frac{\partial \bar{\vartheta}}{\partial x}\vartheta^* &= \bar{\lambda} u^*,\\
-K^* v^* + \left(\frac{\partial \bar{\mathbf{u}}}{\partial y}, \mathbf{u}^*\right) - \Pi v^* - lu^* - \bar{\rho}\frac{\partial p^*}{\partial y} + \frac{\partial \bar{\vartheta}}{\partial y}\vartheta^* &= \bar{\lambda} v^*,\\
-K^* w^* + \left(\frac{\partial \bar{\mathbf{u}}}{\partial z}, \mathbf{u}^*\right) - \Pi w^* - \bar{\rho}\frac{\partial p^*}{\partial z} + \left(\frac{\partial \bar{\vartheta}}{\partial z} + \gamma_a\right)\vartheta^* &= \bar{\lambda} w^*,\\
-\sigma\bar{T}\operatorname{div}\frac{\mathbf{u}^*}{\rho} + \frac{g}{\rho} w^* &= \bar{\lambda}\rho^*,\\
-K^*\vartheta^* - \Pi\vartheta^* - \sigma\bar{p}\operatorname{div}\frac{\mathbf{u}^*}{\rho} &= \lambda\vartheta^*,
\end{aligned}
\tag{13}
$$

where $\mathbf{u} = ui + \bar{u}j + \bar{w}k$, in the case of the adjoint equations. The boundary conditions for the system of basic equations are given in the form (2), and for the system of adjoint equations–in the form (5).

Naturally, the algorithm considered permits various simplifications in the formulation of the problem, for example, adoption of the condition of quasistaticity, neglect of the vertical component of acceleration, etc. The solution of the systems of basic and adjoint equations should be sought in difference form, reducing the spectral problems for differential operators to problems of linear algebra, using the solution of the complete eigenvalue problem.

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Note: Figure translations are in progress. See original paper for figures.

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