ON THE MULTIPLICATIVE CLOSEDNESS OF A SYSTEM OF ELEMENTS OF A RING OF QUOTIENTS

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Abstract

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ON THE MULTIPLICATIVE CLOSEDNESS OF A SYSTEM OF ELEMENTS OF A RING OF QUOTIENTS

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Professor H. J. Weinert drew the author' s attention to the fact that the system of elements $S_{1(S_2)}$ of the ring $R_{(S_2)}$ need not be multiplicatively closed (see ([1])). Thus, the results of ([1]) are valid only under the condition that the system $S_{1(S_2)}$ is multiplicatively closed. Below we shall give a condition that is sufficient for this.*

A multiplicatively closed system S of elements of a ring R, not containing zero (an m.c. system), is called a maximal m.c. system if it is not contained in any other m.c. system. For an m.c. system S, by $\{S, r\}$, where $r \in R \setminus S$, we shall denote the least m.c. system containing S and r, if such a system exists. The elements of the system $\{S, r\}$ have the form $s_0 r s_1 r \dots r s_n$, where $s_i \in S$, and some of the s_i may be absent.

Let S be an m.c. system. A two-sided ideal I of the ring R is called almost S-prime if $I \cap S = \emptyset$ and from $rs \in I$ or $sr \in I$, where $r \in R$ and $s \in S$, it follows that $r \in I$ ([2]). We shall call an m.c. system S maximal with respect to an almost S-prime ideal I if the ideal I is not almost $\{S, r\}$-prime for any $r \in R \setminus S$ ([3]).

A maximal m.c. system S is maximal with respect to any almost S-prime ideal. The converse is not true. In the ring $Z/(6)$, take two m.c. systems $S = \{\overline{1}, \overline{5}\}$ and $S_1 = \{\overline{1}, \overline{2}, \overline{4}, \overline{5}\}$. The ideal 0 is almost S-prime, since there are no zero divisors in S. Any m.c. system $\{S, r\}$, where $r \in R \setminus S$, already contains zero divisors, i.e. 0 is not an almost $\{S, r\}$-prime ideal, and the m.c. system S is maximal with respect to the ideal 0. At the same time $S \subset S_1$, i.e. the m.c. system S is not maximal.

Lemma. *An m.c. system S of a ring R is maximal with respect to an almost S-prime ideal I if and only if the set of all non-zero-divisors in $\varphi(R) \cong R/I$ coincides with $\varphi(S)$ (see ([3])).*

Proof. Let the system S be maximal with respect to the ideal I, and let $\varphi(r)$ be a non-zero-divisor in $\varphi(R)$. Then from $r_1 r \in I$ or $r r_1 \in I$, where $r_1 \in R$, it follows that $r_1 \in I$. Suppose $r \notin S$ and there exists an m.c. system $\{S, r\}$. Since the ideal I is no longer almost $\{S, r\}$-prime, either $\{S, r\} \cap I \neq \emptyset$, or from the relation $s'r \in I$ (or $rs' \in I$), where $s' \in \{S, r\}$, it does not follow that $r \in I$.

Let $\{S, r\} \cap I \neq \emptyset$, i.e. $s_0 r s_1 r \ldots r s_n \in \{S, r\} \cap I$. Since I is an almost S-prime ideal, $r s_1 r \ldots s_{n-1} r \in I$ or $rr' \in I$, where $r' = s_1 r \ldots r s_{n-1}$. By the preceding, $r' \in I$, and again $r s_2 \ldots s_{n-1} r \in I$. Proceeding similarly, we obtain $r \in I$, which contradicts the condition.

In exactly the same way, from $s'r \in I$ or $rs' \in I$ it follows that $r \in I$. Thus I is an almost $\{S, r\}$-prime ideal, which is impossible if $r \in R \setminus S$. Consequently, $\varphi(r) \in \varphi(S)$.

If, however, the m.c. system $\{S, r\}$ does not exist, then $s_0 r s_1 r \ldots s_{n-1} r s_n = 0$, and we can argue as above.

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* Taking this occasion, the author expresses his gratitude to Professor H. J. Weinert.

Let now $\varphi(S)$ be the set of all non-zero-divisors in $\varphi(R)$. If $r \in R \setminus \bar{S}$, then there exists an element $r_1 \in R \setminus I$ such that $rr_1 \in I$ or $r_1 r \in I$. If $r \notin I$, then the second condition in the definition of an almost $\{S, r\}$-prime ideal is violated. If, however, $r \in I$, then $\{S, r\} \cap I = \emptyset$. Thus S is maximal relative to the almost S-prime ideal I. The lemma is proved (see (3)).

Assertion. If $I_1 \supset I_2$, $S_1 \supset S_2$, and the system S_1 is maximal relative to the ideal I_1, then the system $S_{1(S_2)}$ is multiplicatively closed (see (1)).

Proof. Let $\dfrac{s_1}{s_2}, \dfrac{s_1'}{s_2'} \in S_{1(S_2)}$. Then

$$\frac{s_1}{s_2}\frac{s_1'}{s_2'} = \frac{rs_1'}{s_2'\alpha},$$

where $s_1\alpha - s_2' r \in I_2$, $r \in R$, and $\alpha \in S_2$. Since $I_2 \subset I_1$ and $S_2 \subset S_1$, we have $s_1\alpha \in S_1$, $s_2' \in S_1$, and $s - \bar{s}r \in I$, where $s = s_1\alpha$, $\bar{s} = s_2'$. In the ring $\varphi_1(R) \cong R/I_1$ we have the equality $\varphi_1(s) = \varphi(\bar{s})\varphi_1(r)$, and in the ring $R_{(S_1)}$ the element $\varphi_1(r) = \varphi_1(\bar{s})^{-1}\varphi_1(s)$ is invertible, i.e. $\varphi_1(r)$ is a non-zero-divisor in $\varphi_1(R)$. By the lemma, $\varphi_1(r) \in \varphi_1(S_1)$, and $r \in S_1$. Thus $rs_1' \in S_1$ and

$$\frac{s_1}{s_2}\frac{s_1'}{s_2'} \in S_{1(S_2)}.$$

The assertion is proved.

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[3] V. P. Elizarov, A. I. Pilatovskaya, *Fifth All-Union Colloquium on General Algebra* (summary of communications and reports), Novosibirsk, 1963, p. 20.

Note: Figure translations are in progress. See original paper for figures.