SOLUTION OF THE CAUCHY PROBLEM AND OF A MIXED PROBLEM FOR A PARABOLIC SYSTEM

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Abstract

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SOLUTION OF THE CAUCHY PROBLEM AND OF A MIXED PROBLEM FOR A PARABOLIC SYSTEM

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In the note ([1]) a mixed problem for the system (1) of heat and substance transfer ([2]), containing time derivatives in the boundary condition, was solved by the contour-integral method. However, if the boundary condition does not contain time derivatives, then the method of the theory of the heat potential can be applied to the solution of such a problem. In this connection, with the aid of the results of ([1]), in the present note first, by the contour-integral method ([3]), the fundamental matrix of solutions of system (1) is constructed in finite form. With the aid of the constructed fundamental matrix (8) (or (9)) the solution of the Cauchy problem (1), (3) is represented in the form (10). Then the well-known scheme for applying the method of the theory of the heat potential to the solution of the mixed problem (1)—(3) is given. In other words, the solution of problem (1)—(3) for $\Phi(x) \equiv 0$ is represented in the form of a simple-layer potential (11), the density of which is obtained from the corresponding Volterra integral equation (12) by the method of successive approximations.

1. Consider the mixed problem

$$\frac{\partial v}{\partial t} = A\Delta v; \tag{1}$$

$$\lim_{x\to z}\left(\alpha(z,t)\frac{dv(x,t)}{dn_z} + \beta(z,t)v(x,t)\right) = \psi(z,t), \quad z \in \Gamma. \tag{2}$$

$$v(x;0) = \Phi(x), \tag{3}$$

where: 1) A is a constant square matrix of second order, composed of elements a_{ij} $(i,j = 1,2)$, and system (1) is parabolic in the sense of I. G. Petrovskii; 2) $\alpha(z,t)$, $\beta(z,t)$ are square matrices of second order, continuous on the closed Lyapunov surface Γ, which is the boundary of the three-dimensional domain D (this domain, generally speaking, may also be unbounded if an exterior problem is being considered), at points $x = (x_1, x_2, x_3)$, where problem (1)—(3) is considered, $\det \alpha(z,t) \neq 0$ for $z \in \Gamma$, $t \in [0,T]$, $\psi(z,t)$ is a vector function continuous on Γ with respect to $t \in [0,T]$; 3) $\Phi(x)$ is a twice continuously differentiable vector function in the domain D with bounded derivatives there.

2. Consider the system

$$A\Delta u = \lambda^2 u = \Phi(x) \tag{a}$$

with complex parameter λ.

In the note ([1]), under condition 1), a fundamental matrix $P(x,\lambda)$ of solutions of the homogeneous system (a) was constructed in finite form. If the roots p, q of the quadratic equation

$$\mu^2 + (a_{11} + a_{22})\mu + a_{11}a_{22} - a_{12}a_{21} = 0 \tag{6}$$

are distinct, then for the elements $P_{ks}(x,\lambda)$ of the matrix $P(x,\lambda)$ the formulas hold–

of the system (1)

$$P_{11}(x,\lambda) = \frac{1}{4\pi(p-q)|x|}\left\{\frac{a_{22}+p}{p}\exp\left[-\lambda\frac{|x|}{\sqrt{-p}}\right] - \frac{a_{22}+q}{q}\exp\left[-\lambda\frac{|x|}{\sqrt{-q}}\right]\right\},$$

$$P_{ks}(x,\lambda) = \frac{-a_{ks}}{4\pi(p-q)|x|}\left\{\frac{1}{p}\exp\left[-\lambda\frac{|x|}{\sqrt{-p}}\right] - \frac{1}{q}\exp\left[-\lambda\frac{|x|}{\sqrt{-q}}\right]\right\}$$

$$(k \neq s;\ k,s = 1,2), \tag{4}$$

$$P_{22}(x,\lambda) = \frac{1}{4\pi(p-q)|x|}\left\{\frac{a_{11}+p}{p}\exp\left[-\lambda\frac{|x|}{\sqrt{-p}}\right] - \frac{a_{11}+q}{q}\exp\left[-\lambda\frac{|x|}{\sqrt{-q}}\right]\right\}.$$

If the roots p, q of equation (6) coincide ($p = q$), then $P_{ks}(x,\lambda)$ are determined by the formulas

$$P_{11}(x,\lambda) = \frac{-1}{8\pi p^2}\left\{\lambda\sqrt{-p}\left(1 + \frac{a_{22}}{p}\right) + \frac{2a_{22}}{|x|}\right\}\exp\left[-\lambda\frac{|x|}{\sqrt{-p}}\right],$$

$$P_{k,s}(x,\lambda) = \frac{a_{ks}}{8\pi p^2}\left\{\frac{\lambda\sqrt{-p}}{p} + \frac{2}{|x|}\right\}\exp\left[-\lambda\frac{|x|}{\sqrt{-p}}\right] \quad (k \neq s;\ k,s = 1,2), \tag{5}$$

$$P_{22}(x,\lambda) = \frac{-1}{8\pi p^2}\left\{\lambda\sqrt{-p}\left(1+\frac{a_{11}}{p}\right)+\frac{2a_{11}}{|x|}\right\}\exp\left[-\lambda\frac{|x|}{\sqrt{-p}}\right].$$

Let S be an infinite open contour in the λ-plane, coinciding outside a circle of sufficiently large radius with center at the origin with continuations of the rays $\cos\arg\lambda = \delta$, where δ is a sufficiently small positive number. By the method of the contour integral [3] it is easily proved that

Theorem 1. *The function defined by the formula*

$$v_1(x,t) = \frac{-1}{\pi\sqrt{-1}}\int_S e^{\lambda^2 t}\lambda\, d\lambda \int_D P(x-\xi,\lambda)\Phi(\xi)\, dD_\xi, \tag{6}$$

is a solution of the Cauchy problem (1), (3) in the domain D.

Interchanging the order of integration with respect to λ and ξ, from (6) we obtain

$$v_1(x,t) = \int_D \left\{\frac{-1}{\pi\sqrt{-1}}\int_S e^{\lambda^2 t}\lambda P(x-\xi,\lambda)\, d\lambda\right\}\Phi(\xi)\, dD_\xi.$$

Consequently, the matrix

$$Q(x-\xi,t) = \frac{-1}{\pi\sqrt{-1}}\int_S e^{\lambda^2 t}\lambda P(x-\xi,\lambda)\, d\lambda \tag{7}$$

is the fundamental matrix of solutions of system (1).

It should be noted that the contour integral (7) is computed [3], and the fundamental matrix $Q(x,t)$ of system (1) is constructed in closed form. In other words, the following holds.

Theorem 2. *Under condition 1), if the roots p, q of equation (6) are distinct, then system (1) has a fundamental matrix $Q(x,t)$ of solutions, whose elements $Q_{ks}(x,t)$ are determined by the formulas*

$$Q_{11}(x,t) = \frac{-1}{8\pi^{3/2}(p-q)}\left\{\frac{a_{22}+p}{(\sqrt{-p})^3}\exp\frac{|x|^2}{4pt} - \frac{a_{22}+q}{(\sqrt{-q})^3}\exp\frac{|x|^2}{4qt}\right\},$$

$$Q_{ks}(x,t) = \frac{a_{ks}}{(2\sqrt{\pi t})^3(p-q)}\left\{\frac{1}{(\sqrt{-p})^3}\exp\frac{|x|^2}{4pt} - \frac{1}{(\sqrt{-q})^3}\exp\frac{|x|^2}{4qt}\right\}$$

$$(k \neq s;\ k,s = 1,2), \tag{8}$$

$$Q_{22}(x,t) = \frac{-1}{(2\sqrt{\pi t})^3(p-q)} \left\{ \frac{a_{11}+p}{(\sqrt{-p})^3} \exp \frac{|x|^2}{4pt} - \frac{a_{11}+q}{(\sqrt{-q})^3} \exp \frac{|x|^2}{4qt} \right\}.$$

If the roots p, q of equation (6) coincide ($p = q$), then for $Q_{sk}(x,t)$ the following formulas hold

$$Q_{11}(x,t) = \frac{-1}{(2\sqrt{-p\pi t})} \left\{ \left(1 + \frac{a_{22}}{p}\right) \frac{|x|^2}{4pt} + 1 + \frac{3a_{22}}{2p} \right\} \exp \frac{|x|^2}{4pt},$$

$$Q_{22}(x,t) = \frac{-1}{(2\sqrt{-p\pi t})} \left\{ \left(1 + \frac{a_{11}}{p}\right) \frac{|x|^2}{4pt} + 1 + \frac{3a_{11}}{2p} \right\} \exp \frac{|x|^2}{4pt}, \tag{9}$$

$$Q_{ks}(x,t) = \frac{a_{ks}}{p(2\sqrt{-p\pi t})} \left(\frac{|x|^2}{4pt} + \frac{3}{2} \right) \exp \frac{|x|^2}{4pt} \qquad (k \neq s;\ k,s = 1,2).$$

From Theorem 2 it follows:

Theorem 3. *Under condition 1), there exists a solution $v_1(x,t)$ of the Cauchy problem (1), (3), representable by the formula*

$$v_1(x,t) = -\int_D Q(x-\xi,t)\Phi(\xi)\, dD_\xi. \tag{10}$$

Obviously, with the aid of (10), problem (1)–(3) can be reduced to an analogous problem for which $\Phi(x) \equiv 0$. The following holds:

Theorem 4. *Under conditions 1)–2), problem (1)–(3) for $\Phi(x) \equiv 0$ has a solution $v_2(x,t)$, representable in the form of a simple-layer potential*

$$v_2(x,t) = \int_0^t d\tau \int_\Gamma Q(x-y,t-\tau)\mu(y,\tau)\, d\Gamma_y, \tag{11}$$

whose density is determined by the method of successive approximations from a Volterra-type integral equation

$$\mu(z,t) = 2\alpha^{-1}(z,t)\psi(z,t)+$$

$$+\int_0^t \int_\Gamma \alpha^{-1}(z,t) \left(\frac{d}{dn_z} + \beta(z) \right) Q(z-y,t-\tau)\mu(y,\tau)\, d\tau\, d\Gamma_y. \tag{12}$$

Remark. By the indicated scheme one can also solve the mixed problem for system (1) with a nonlinear boundary condition.

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Note: Figure translations are in progress. See original paper for figures.