ON THE CONSTRUCTION OF PROPER GUIDING FUNCTIONS

MATHEMATICS

1968

SovietRxiv

Abstract

Full Text

UDC 517.91:517.53

MATHEMATICS

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ON THE CONSTRUCTION OF PROPER GUIDING FUNCTIONS

(Presented by Academician I. G. Petrovskii on 21 III 1968)

In recent years M. A. Krasnosel' skii and his students have developed a new method for proving the existence of bounded and periodic solutions of systems of ordinary differential equations and of systems of equations with distributed arguments. In the case of systems of ordinary differential equations the main role is played by the presence of so-called guiding functions, in many respects analogous to Lyapunov functions in stability theory. In the present article a class of systems of second-order differential equations is described for which guiding functions can be constructed. As one of the consequences we obtain a generalization of the well-known theorems of Gomory ([2,3]) on periodic solutions of second-order systems whose principal parts of the right-hand sides are homogeneous forms.

1. A continuously differentiable function $V(x, y)$ is called nondegenerate if its gradient vanishes only inside some ball; the index $\gamma(V)$ of a nondegenerate function $V(x, y)$ is the rotation of the vector field $\operatorname{grad} V$ on the circles $x^2 + y^2 = R^2$ of large radii R.

Consider the system

$$dx/dt = f(x, y, t), \qquad dy/dt = g(x, y, t). \tag{1}$$

A nondegenerate function $V(x, y)$ is called a proper guiding function for system (1) if, for $x^2 + y^2 \geq R_0^2$, the inequality

$$V'_x(x, y)f(x, y, t) + V'_y(x, y)g(x, y, t) \geq$$

$$\geq a_0\sqrt{[V'_x(x, y)]^2 + [V'_y(x, y)]^2}\sqrt{f^2(x, y, t) + g^2(x, y, t)}, \tag{2}$$

is satisfied, where $a_0 > 0$, and if there exists a continuously differentiable function $W(x, y)$ such that

$$[W'_x(x,y)]^2 + [W'_y(x,y)]^2 \leq [V'_x(x,y)]^2 + [V'_y(x,y)]^2 \quad (x^2+y^2 \geq R_0^2), \tag{3}$$

$$\lim_{x^2+y^2\to\infty} W(x,y) = \infty. \tag{4}$$

Let us emphasize that the proper guiding function $V(x,y)$ itself need not even have a definite sign.

The importance of the concept of a proper guiding function is determined, in particular, by the following fundamental theorem of M. A. Krasnosel' skii.

Theorem 1 ([1]). *Suppose that for system (1) a proper guiding function can be constructed whose index is different from zero. Then system (1) has at least one uniformly bounded solution defined on the whole axis. If the right-hand sides of system (1) are ω-periodic in t, then system (1) has at least one ω-periodic solution.*

2. In what follows we shall assume that system (1) has the form

$$dx/dt = F(x,y) + f_1(x,y,t), \qquad dy/dt = G(x,y) + g_1(x,y,t), \tag{5}$$

where F and G are continuously differentiable homogeneous functions of order of degree n $\left(F(\lambda x, \lambda y) \equiv \lambda^{nF}(x,y),\ G(\lambda x,\lambda y) \equiv \lambda^{nG}(x,y)\right)$ and, uniformly with respect to t,

$$\lim_{x^2+y^2\to\infty} \frac{f_1(x,y,t)}{(x^2+y^2)^{n/2}} = \lim_{x^2+y^2\to\infty} \frac{g_1(x,y,t)}{(x^2+y^2)^{n/2}} = 0. \tag{6}$$

We shall also assume that the vector field

$$P(x,y) = \{F(x,y), G(x,y)\} \tag{7}$$

is nondegenerate on the circles $x^2+y^2 = R^2$, and denote by γ_0 its rotation on the unit circle.

Theorem 2. *Let $\gamma_0 \geqslant 2$. Then a regular guiding function cannot be constructed for system (5).*

We note that for the homogeneous system (5) even an ordinary guiding function cannot be constructed (for the definition see ([1])).

3. Consider in the plane $\{x,y\}$ a polar coordinate system $\{\varphi,\rho\}$, and denote by $\theta(\varphi)$ a continuous angular function of the field (7), defined on the interval $0 \leqslant \varphi < \infty$, specified on the unit circle. Let the angle between the vector of the field (7) at the point $\varphi = 0$ and the polar axis be equal to φ_0. In what follows we shall assume that the equation $\theta'(\varphi) = 1$ has a finite number of roots on the interval $[0, 2\pi]$.

We shall say that the field (7) is quasi-monotone if between any two roots of the equation $\theta(\varphi) - \varphi + \varphi_0 = k\pi$ there are no roots of the equation $\theta(\varphi) - \varphi + \varphi_0 = (k+1)\pi$.

Theorem 3. *Let $\gamma \leqslant 0$. Then a regular guiding function for system (5) exists if and only if the field (7) is quasi-monotone.*

It is clear that the index $\gamma(V)$ of the guiding function $V(x, y)$ existing under the conditions of Theorem 3 coincides with γ_0. Therefore Theorem 3, together with Theorem 1, gives criteria for the existence of bounded and periodic solutions of system (5). For example, the following is true.

Theorem 4. *Let $\gamma_0 < 0$ and let the field (7) be quasi-monotone. Then system (5) has at least one solution bounded on the whole axis. If the functions $f_1(x, y, t)$ and $g_1(x, y, t)$ are ω-periodic in t, then system (5) has at least one ω-periodic solution.*

Theorem 4 contains Gomory' s criterion ([2]) (see also ([3])) for the existence of periodic solutions of systems of the form (5), in which $F(x, y)$ and $G(x, y)$ are homogeneous polynomials in the variables x and y.

4. More complicated is the case when $\gamma_0 = 1$. A vector $\{x_0, y_0\}$ $(x_0^2 + y_0^2 = 1)$ is called an eigenvector of the field (7) if, for some nonzero λ_0, $F(x_0, y_0) = \lambda_0 x_0$, $G(x_0, y_0) = \lambda_0 y_0$.

Theorem 5. *Let $\gamma_0 = 1$. Suppose the field (7) has at least one eigenvector. Then a regular guiding function for system (5) exists if and only if the field (7) is quasi-monotone.*

Theorem 6. *Let $\gamma_0 = 1$. Suppose the field (7) has no eigenvectors. Then a regular guiding function for system (5) exists if and only if*

$$\int_0^{2\pi} \operatorname{ctg}[\theta(\varphi) - \varphi]\, d\varphi \neq 0. \tag{11}$$

It is easy to indicate examples of fields for which all four possible variants of the application of Theorems 5 and 6 are realized.

The author expresses his gratitude to M. A. Krasnosel'skii for posing the problem and for his guidance.

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Received
13 III 1968

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Note: Figure translations are in progress. See original paper for figures.