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Abstract

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ON THE RESOLVENT OF GENERALIZED SELF-ADJOINT EXTENSIONS OF SYMMETRIC OPERATORS

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In the present note a number of theorems on the resolvent of generalized self-adjoint extensions are considered.

I. Consider a triple of spaces $G_+ \subseteq G_0 \subseteq G_-$, and let B be an arbitrary generalized self-adjoint operator acting from G_+ into G_-. We shall regard it as an operator in the space G_- with dense domain G_+. Note that in the space G_- the operator B is, generally speaking, nonsymmetric. It is easy to see that all eigenvalues of the operator B ($Bf = \lambda f,\ f \in G_+$) are real. Denote $\Delta_B(\lambda) = (B - \lambda I)G_+$.

Theorem 1. *The number λ is an eigenvalue of the generalized self-adjoint operator $B(G_+ \to G_-)$ if and only if*

$$\overline{\Delta_B(\bar{\lambda})} \neq G_-.$$

Proof. Let λ be an eigenvalue of B, so that $Bf = \lambda f$ ($f \neq 0$). In that case, for any $g \in G_+$,

$$(J^{-1}f, (B - \lambda I)g)_- = (f, J(B - \lambda I)g)_+ =$$
$$= (f, (B - \lambda I)g)_0 = (Bf - \bar{\lambda} f, g)_0 = 0,$$

and, consequently, the vector $J^{-1}f \neq 0$ is orthogonal to $\Delta_B(\lambda)$, which is possible only when

$$\overline{\Delta_B(\lambda)} \neq G_-.$$

Now suppose that

$$\overline{\Delta_B(\bar{\lambda})} \neq G_-.$$

In that case there exists a vector $\alpha \in G_-$ orthogonal to the manifold $\Delta_B(\lambda)$. Therefore, for any $g \in G_+$,

$$(\alpha, (B - \lambda I)g)_- = (J\alpha, J(B - \lambda I)g)_+ =$$

$$= (J\alpha, (B - \lambda I)g)_0 = ((B - \overline{\lambda} I)J\alpha, g)_0 = 0.$$

It follows from this that the vector $f = J\alpha \neq 0$ ($f \in G_+$) is an eigenvector of the operator B, corresponding to the eigenvalue λ ($\lambda = \overline{\lambda}$). The theorem is proved.

Corollary. *If λ is a nonreal number, then*

$$\overline{\Delta_B(\overline{\lambda})} = G_-.$$

II. Let A be a symmetric operator with defect index (r, r) ($r < \infty$), acting in G_0. Consider the Hilbert space $G_+ = D_{A^*}$ with scalar product

$$(f, g)_+ = (A^* f, A^* g)_0 + (f, g)_0 \qquad (f, g \in D_{A^*})$$

and construct, as was done in ([1,3]), a triple of spaces $G_+ \subseteq G_0 \subseteq G_-$. In ([5]) it was shown that A can be extended to $G_+ = D_{A^*}$ in such a way that the resulting extension A_{G_+} ($G_+ \to G_-$) is a generalized self-adjoint operator.

Theorem 2. *If λ is a nonreal number and A_{G_+} is an arbitrary generalized self-adjoint extension of a symmetric operator with defect index (r, r) ($r < \infty$), then $R_\lambda = (A_{G_+} - \lambda I)^{-1}$ continuously maps*

maps the Hilbert space G_- onto the Hilbert space G_0, and, moreover,

$$\|R_\lambda\| \leqslant \frac{\sqrt{2}}{\sin \varphi_\lambda}(1 + |\lambda|)\left(1 + \frac{1}{|\operatorname{Im} \lambda|}\right) \quad (\varphi_\lambda \neq 0), \tag{1}$$

where φ_λ is the minimal angle between certain subspaces.

We outline the proof of this theorem. Denote

$$\Delta_{A_{G_+}}(\lambda) = (A_{G_+} - \lambda I)G_+.$$

Then, if $a = (A_{G_+} - \lambda I)f$ ($f \in G_+$, $a \in G_-$), then

$$\|a\|_- \geqslant |\operatorname{Im} \lambda| \, \|f\|_0^2 / \|f\|_+. \tag{2}$$

Further,

$$\|a\|_- = \sup_{\varphi \in G_+} \frac{|(\varphi, a)_0|}{\|\varphi\|_+} = \sup_{\varphi \in G_+} \frac{|(\varphi, (A_{G_+} - \lambda I)f)_0|}{\|\varphi\|_+} \geqslant \sup_{\psi \in D_A} \frac{|((A - \bar{\lambda} I)\psi, f)_0|}{\|A\psi\|_0 + \|\psi\|_0}. \tag{3}$$

It follows that

$$\|a\|_- \geqslant \sup_{\psi \in D_A} \frac{|((A - \bar{\lambda} I)\psi, f)_0|}{(1 + |\lambda|)\left(1 + \frac{1}{|\operatorname{Im} \lambda|}\right) \|(A - \bar{\lambda} I)\psi\|_0}.$$

Denote

$$\Delta_A(\bar{\lambda}) = (A - \bar{\lambda} I) D_A.$$

Obviously, $\Delta_A(\bar{\lambda})$ is a subspace in G_0. Then

$$G_0 = \Delta_A(\bar{\lambda}) + \Delta_A(\lambda)^{\perp}. \tag{4}$$

As is known, $\Delta_A(\bar{\lambda}) = \mathfrak{N}_\lambda$, where $\mathfrak{N}_\lambda$ is the eigenspace of the operator A^* corresponding to the eigenvalue λ. From (2) and (3) it follows that

$$\|a\|_- \geqslant \frac{1}{(1+|\lambda|)(1+1/|\operatorname{Im}\lambda|)} \|Pf\|_0, \tag{5}$$

where P is the projection operator onto $\Delta_A(\bar{\lambda})$. From relations (2), (3), (4), and (5) it follows that

$$\|(A_{G_+} - \lambda I)Pf\|_- \geqslant \frac{1}{(1+|\lambda|)(1+1/|\operatorname{Im}\lambda|)} \|Pf\|_0; \tag{6}$$

$$\|(A_{G_+} - \lambda I)Qf\|_- \geqslant \frac{|\operatorname{Im}\lambda|}{\sqrt{|\lambda|^2+1}} \|Qf\|_0. \tag{7}$$

Here Q is the projection operator onto $\mathfrak{N}_\lambda$. Denote

$$G_-^{(1)} = (A_{G_+} - \lambda I)Pf, \qquad G_-^{(2)} = (A_{G_+} - \lambda I)Qf \quad (f \in G_+).$$

Then from (4)

$$\Delta_{A_{G_+}}(\lambda) = G_-^{(1)} + G_-^{(2)}.$$

Denote by $\cos\varphi_\lambda$ the cosine of the minimal angle ([2]) between the closures in G_- of the linear manifolds $G_-^{(1)}$ and $G_-^{(2)}$; taking into account relations (6) and (7), we obtain

$$\|(A_{G_+} - \lambda I)f\|_-^2 \geqslant \frac{1-\cos^2\varphi_\lambda}{2} \frac{1}{(1+|\lambda|)^2\left(1+\dfrac{1}{|\operatorname{Im}\lambda|}\right)^2} \|f\|_0^2.$$

Since, by virtue of the corollary to Theorem 1, $\overline{\Delta}_{A_{G_+}}(\lambda) = G_-$, relation (1) follows from the last relation.

Theorem 3. Every generalized self-adjoint extension A_G of a symmetric operator A with defect index (r, r) $(r < \infty)$ does not admit a closure as an operator acting from G_0 into G_-.*

* Consequently, a fortiori, A_{G_+} does not admit a closure as an operator acting in G_- and having dense domain of definition G_+.

Proof. We shall show that there exists a sequence f_n ($f_n \in G_+$, $n = 1, 2, ...$), converging to zero in the metric G_0, and such that $A_{G_+} f_n$ converges in the metric G_-, with

$$\lim_{n\to\infty} A_{G_+} f_n \neq 0.$$

In (5) it was shown that every generalized self-adjoint extension A_{G_+} of the operator A has the form

$$A_{G_+} f = A^* f + \sum_{k,j=1}^{r} [a_{jk}(f, \hat{e}_j)_0 + b_{jk}(f, \hat{q}_j)_0] \hat{g}_k + \sum_{k,j=1}^{r} [c_{jk}(f, \hat{e}_j)_0 + d_{jk}(f \hat{g}_j)_0] \hat{e}_k,$$

where the coefficient matrices satisfy the relations

$$D = A^*, \qquad c_{kj} = \bar{c}_{jk}, \qquad b_{kj} = \bar{b}_{jk} \quad (k \neq j),$$

$$\operatorname{Im} c_{jj} = -\frac{1}{2}, \qquad \operatorname{Im} b_{jj} = \frac{1}{2}.$$

Let

$$f_n = f_A^{(n)} + e_j,$$

where, in the metric G_0,

$$f_A^{(n)} \to (-e_j), \qquad f_A^{(n)} \in D_A$$

and

$$e_j \in \mathfrak{N}_i, \qquad e_j = J \hat{e}_j$$

(see (5)). Thus $f_n \to 0$ in the metric G_0. It can be shown that

$$\alpha_n = A_{G_+} f_n$$

will converge in the metric G_-.

Further,

$$(\alpha_n, g)_0 = (f_A^{(n)}, A^* g)_0 = i(e_j, g)_0 + \sum_{k=1}^{r} a_{jk} (\hat{g}_k, g)_0 + \sum_{k=1}^{r} c_{jk} (\hat{e}_k, g)_0.$$

Let

$$\alpha = \lim_{n\to\infty} \alpha_n.$$

Then

$$(\alpha, g)_0 = i(e_j, g)_0 - (e_j, A_g^* g)_0 + \sum_{k=1}^{r} a_{jk} (\hat{g}_k, g)_0 + \sum_{k=1}^{r} c_{jk} (\hat{e}_k, g)_0.$$

Analyzing the last relation, one can establish,* that

$$\|\alpha\|_- = \sup_{\|g\|_+ \le 1} |(\alpha, g)_0| \neq 0.$$

Thus, we have indicated a sequence $f_n \in G_+$ which tends to zero in the metric G_0, while $\alpha_n = A_{G_+} f_n$ does not tend to zero in the metric G_-.

Theorem 4*. Let $\widetilde{A}$ be a self-adjoint extension of a symmetric operator A, and let A_{G_+} be a generalized self-adjoint extension of the operator A which is also an extension of $\widetilde{A}$. Then, if λ is a regular point for $\widetilde{A}$, then

$$R_\lambda = (A_{G_+} - \lambda I)^{-1}$$

maps G_- continuously into G_0.

We outline the proof of this theorem. Denote

$$\|\alpha\| = \sup \frac{|(\varphi, (A_{G_+} - \lambda I) f)_0|}{\|\varphi\|_+} \geq \sup_{\psi \in D_{\widetilde{A}}} \frac{|((\widetilde{A} - \overline{\lambda} I)\psi, f)_0|}{\|\widetilde{A}\psi\|_0 + \|\psi\|_0} \geq$$

$$\geq \sup_{\psi \in D_\sim} \frac{|((\widetilde{A} - \overline{\lambda} I)\psi, f)_0|}{(1 + |\lambda|)(1 + \|(\widetilde{A} - \overline{\lambda} I)^{-1}\|_0)\|(\widetilde{A} - \overline{\lambda} I)\psi\|_0} =$$

$$= \frac{1}{(1 + |\lambda|)(1 + \|(\widetilde{A} - \overline{\lambda} I)^{-1}\|_0)} \|f\|_0.$$

* In this theorem it is not required that the operator A have finite defect numbers.

It can be shown that if f is an arbitrary vector from G_0, then in the range of values of $(A_{G_+} - \lambda I)$ there exists a sequence α_n, convergent in G_-, for which $(A_{G_+} - \lambda I)^{-1}\alpha_n \to f$ in the metric of G_0. Hence, and from the indicated inequalities, the assertion of the theorem follows.

Let us note, in conclusion, that theorems analogous to Theorems 2 and 3 can also be obtained for operators with unequal deficiency indices.

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Note: Figure translations are in progress. See original paper for figures.