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Abstract

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SOME VARIATIONAL PROBLEMS WITH ACCOUNT TAKEN OF THE COST OF CONTROL IN HILBERT SPACES

1. Denote by H a real Hilbert space with scalar product (,) and norm $\| \ \|$. The norms and scalar products of the Hilbert spaces H_1, H_2, and H_3 will be denoted by the corresponding subscript. Let $\mathbf{L}_2 = \mathbf{L}_2(H, [0, T])$ be the Hilbert space of vector-functions with values in H (see (1)).

Theorem. Let $D(L_2 \to L_2)$, $F(H \to L_2)$, $B(L_2 \to H_3)$, and $C(H \to H_3)$ be linear bounded operators. Then for any prescribed pairs $\xi_1 \in H_3$, $\xi_2(t) \in L_2$ there exists a pair $\eta \in L_2$, $\zeta \in H$, minimizing the functional

$$E(\eta, \zeta) = \|B\eta + C\zeta - \xi_1\|_3^2 + \int_0^T \|D\eta + F\zeta - \xi_2\|^2 \, dt + \alpha \int_0^T \|\eta\|^2 \, dt + \gamma \|\zeta\|^2, \quad (1)$$

and η and ζ are determined uniquely from the system of equations

$$(B^*B + D^*D + \alpha)\eta + (B^*C + D^*F)\zeta = B^*\xi_1 + D^*\xi_2,$$

$$(C^*B + F^*D)\eta + (C^*C + F^*F + \gamma)\zeta = C^*\xi_1 + F^*\xi_2. \quad (2)$$

Here and throughout the sequel $\alpha, \gamma > 0$.

For the proof of the theorem one uses the result from (2), if for H_1 we take $L_2 \times H$ with scalar product of the elements $x = \langle \eta, \zeta \rangle$ and $y = \langle \psi, \varphi \rangle$, defined by the formula $(x, y)_1 = \alpha[\eta, \psi] + \gamma(\zeta, \varphi)$, and in $H_2 = H_1 \times L_2$ introduce the scalar product of the elements $z = \langle\langle \xi, \eta \rangle\rangle$ and $r = \langle\langle \tau, \psi \rangle\rangle$ by the formula $(z, r)_2 = (\xi, \tau)_3 + [\eta, \psi]$, and as the operator $K(H_1 \to H_2)$ from (2) take the operator $Kx = \langle\langle B\eta + C\zeta, D\eta + F\zeta \rangle\rangle$.

Then

$$K^*z = \langle \alpha^{-1}(B^*\xi + D^*\eta), \gamma^{-1}(C^*\xi + F^*)\rangle.$$

2. We now consider the Cauchy problem for a differential equation in the Hilbert space H

$$u'(t) + Au(t) = f(t), \qquad u(0) = \varphi, \qquad 0 \le t \le T, \tag{3}$$

where A is a linear bounded self-adjoint operator in H.

In the paper $(^2)$ (see also $(^3)$) the problem was considered of finding a pair $\langle f, \varphi\rangle \in H_1$ minimizing the functionals

$$E_1(\langle f, \varphi\rangle) = \|u(T) - \psi\|^2 + \alpha \int_0^T \|f(t)\|^2\, dt + \gamma\|\varphi\|^2,$$

$$E_2(\langle f, \varphi\rangle) = \int_0^T \|u(t) - g(t)\|^2\, dt + \alpha \int_0^T \|f(t)\|^2\, dt + \gamma\|\varphi\|^2,$$

where $\psi \in H$ and $g(t) \in L_2$ are prescribed elements, and $u(t)$ is the solution of the Cauchy problem (3) with unbounded self-adjoint linear operator A.

We now consider analogous problems, where the "costs of control" remain the same, while into the "cost of deviation"we introduce the derivative of the solution $u'(t)$. More precisely, the following is considered

Problem 1. Find a pair $\langle f, \varphi\rangle \in H_1$ minimizing the functional

$$E_1(\langle f, \varphi\rangle) = \|u(T) - \psi\|^2 + \int_0^T \|u'(t) - h(t)\|^2\, dt + \alpha \int_0^T \|f(t)\|^2\, dt + \gamma\|\varphi\|^2, \tag{4}$$

where $u(t)$ is the solution of problem (3), and $\psi \in H$ and $h(t) \in L_2$ are given elements. The problem has a solution, and a unique one, if in the theorem we put $H_3 = H$ and

$$Bf = \int_0^T e^{-(T-s)A} f(s)\, ds, \qquad C\varphi = e^{-TA}\varphi,$$

$$Df = f(t) - A\int_0^t e^{-(t-\tau)A} f(\tau)\, d\tau, \qquad F\varphi = -Ae^{-tA}\varphi. \tag{5}$$

It is not difficult, in particular, to establish that

$$D^*Df = f(t) - \frac{1}{2}A\int_0^T \left[e^{-(2T-t-\tau)A} + e^{-|t-\tau|A}\right] f(\tau)\,d\tau,$$

$$F^*Df = -\frac{1}{2}A\int_0^T \left[e^{-(2T-\tau)A} + e^{-\tau A}\right] f(\tau)\,d\tau,$$

$$D^*F\varphi = -\frac{1}{2}A\left(e^{-(2T-t)A} + e^{-tA}\right)\varphi, \qquad F^*F\varphi = \frac{1}{2}A\left(I - e^{-2TA}\right)\varphi,$$

$$B^*Bf = \int_0^T e^{-(2T-t-\tau)A} f(\tau)\,d\tau, \qquad C^*Bf = \int_0^T e^{-(2T-\tau)A} f(\tau)\,d\tau.$$

Then system (2), after some transformations, takes the form

$$(1+\alpha)f(t) + \int_0^T W(t,\tau)f(\tau)\,d\tau + W(t,0)\varphi = G_1(t),$$

$$(A+\gamma I)\varphi + \int_0^T W(0,\tau)f(\tau)\,d\tau + W(0,0)\varphi = G_1(0) - h(0), \tag{6}$$

where

$$W(t,\tau) = \left(I - \frac{1}{2}A\right)e^{-(2T-t-\tau)A} - \frac{1}{2}Ae^{-|t-\tau|A},$$

$$G_1(t) = h(t) + e^{-(T-t)A}\psi - A\int_0^T e^{-(\tau-t)A} f(\tau)\,d\tau.$$

From (6), for $t=0$, we obtain

$$\varphi = (A+\gamma I)^{-1}[(1+\alpha)f(0) - h(0)]. \tag{7}$$

Therefore system (6) is replaced by the loaded integral equation

$$(1+\alpha)f(t) + \int_0^T W(t,\tau)f(\tau)\,d\tau + (1+\alpha)W(t,0)(A+\gamma I)^{-1}f(0) =$$

$$= G_1(t) + W(t,0)(A+\gamma I)^{-1}h(0). \tag{8}$$

Problem 2. Let $h_1(t)$ and $h_2(t)$ be given functions from L_2. Find a pair $\langle f,\varphi\rangle \in H_1$ minimizing the functional

$$E_2(\langle f,\varphi\rangle) = \int_0^T \|u(t) - h_1(t)\|^2\,dt + \int_0^T \|u'(t) - h_2(t)\|^2\,dt + \alpha \int_0^T \|f(t)\|^2\,dt + \gamma\|\varphi\|^2, \tag{9}$$

where $u(t)$ is the solution of the Cauchy problem (3).

Functional (9), as in problem 1, is a special case of functional (1), if in the theorem we put $H_3 = L_2$, define the operators $D(L_2 \to L_2)$, $F(H \to L_2)$ as in (5), and define the operators $B(L_2 \to L_2)$ and $C(H \to L_2)$ as follows:

so that

$$Bf = \int_0^t e^{-(t-\tau)A} f(\tau)\,d\tau, \qquad C\varphi = e^{-tA}\varphi.$$

Then system (2) takes the form

$$\begin{aligned}
(1+\alpha)f(t) + \int_t^T \int_0^s e^{-(2s-\tau-t)A} f(\tau)\,d\tau\,ds + \int_0^T \Psi(t,\tau)f(\tau)\,d\tau\, + \\
+ \int_t^T e^{-(2\tau-t)A}\varphi\,d\tau + \Psi(t,0)\varphi = G_2(t), \\
(A+\gamma I)\varphi + \int_0^T \int_0^s e^{-(2s-\tau)A} f(\tau)\,d\tau\,ds + \int_0^T \Psi(0,\tau)f(\tau)\,d\tau\, + \\
+ \int_0^T e^{-2\tau A}\varphi\,d\tau + \Psi(0,0)\varphi = G_2(0) - h_2(0),
\end{aligned} \tag{10}$$

where

$$\Psi(t,\tau) = W(t,\tau) - e^{-(2T-t-\tau)A}, \qquad G_2(t) = h_2(t) + \int_t^T e^{-(\tau-t)A}(h_1(\tau) - Ah_2(\tau))\,d\tau.$$

This system is equivalent to the equation

$$\begin{aligned}(1+\alpha)f(t) + \int_t^T \int_0^s e^{-(2s-\tau-t)A} f(\tau)\,d\tau\,ds \,+ \\ &+ \int_0^T \Psi(t,\tau)f(\tau)\,d\tau + (1+\alpha)(A+\gamma I)^{-1}\int_t^T e^{-(2\tau-t)A} f(0)\,d\tau \,+ \\ &+ (1+\alpha)\Psi(t,0)(A+\gamma I)^{-1}f(0) = \\ &= G_2(t) + \Psi(t,0)(A+\gamma I)^{-1}h_2(0) + (A+\gamma I)^{-1}\int_t^T e^{-(2\tau-t)A} A h_2(0)\,d\tau.\end{aligned} \tag{11}$$

3. Let us present one of the possible realizations of these variational problems. Let $K(x,y) = R(x)R(y)$ be a continuous function for $0 \le x, y \le 1$, and

$$\int_0^1 R^2(\xi)\,d\xi = q < +\infty.$$

If, for each fixed t, for functions $u(t,x) \in L_2[0,1]$ one defines the operator A by the formula

$$Au = (Au)(t,x) = \int_0^1 R(x)R(y)u(t,y)\,dy, \tag{*}$$

then A will be a linear bounded self-adjoint operator mapping $L_2[0,1]$ into itself.

Let a certain process be described by the Cauchy problem for the integro-differential equation

$$\frac{\partial u(t,x)}{\partial t} + \int_0^1 R(x)R(y)u(t,y)\,dy = f(t,x), \qquad u(0,x) = \varphi(x), \quad t \ge 0.$$

Here $H = L_2[0,1]$, the operator A is defined by formula (*) and, for any $v \in H = L_2[0,1]$,

$$e^{-tA}v = v + \frac{e^{-tq}-1}{q}Av, \qquad (A+\gamma I)^{-1}v = \frac{1}{\gamma}v - \frac{1}{\gamma(\gamma+q)}Av.$$

The functionals E_1 and E_2 in this case take the form

$$E_1(\langle f,\varphi\rangle) = \int_0^1 |u(T,x)-\psi(x)|^2\,dx + \int_0^T \int_0^1 \left|\frac{\partial u(t,x)}{\partial t} - h(t,x)\right|^2 dx\,dt +$$

$$+\alpha \int_0^T \int_0^1 |f(t,x)|^2 \, dx \, dt + \gamma \int_0^1 |\varphi(x)|^2 \, dx,$$

$$E_2(\langle f, \varphi \rangle) = \int_0^T \int_0^1 |u(t,x) - h_1(t,x)|^2 \, dx \, dt + \int_0^T \int_0^1 \left| \frac{\partial u(t,x)}{\partial t} - h_2(t,x) \right|^2 dx \, dt$$
$$+ \alpha \int_0^T \int_0^1 |f(t,x)|^2 \, dx \, dt + \gamma \int_0^1 |\varphi(x)|^2 \, dx.$$

Problem 1 becomes the problem of finding an optimal pair $\langle f, \varphi \rangle$ that realizes the best (in a certain sense) approximation of the final result $u(T,x)$ of the process $u(t,x)$ to a prescribed function $\psi(x)$. Its solution is found from the equations

$$\begin{aligned}(1+\alpha) f(t,x) &+ \int_0^T \int_0^1 \Phi(t,x,\tau,y,\tau) f(\tau,y) \, dy \, d\tau \\ &+ (1+\alpha) \int_0^1 P(t,x,y) f(0,y) \, dy \\ &+ \int_0^T \left\{ \varepsilon(t,\tau) f(\tau,x) + \frac{1+\alpha}{\gamma T} \varepsilon(t,0) f(0,x) \right\} d\tau = G(t,x),\end{aligned} \tag{12}$$

$$\varphi(x) = \frac{1}{\gamma}[(1+\alpha) f(0,x) - h(0,x)] - \frac{1}{\gamma(\gamma+q)} \int_0^1 [(1+\alpha) f(0,y) - h(0,y)] R(x) R(y) \, dy, \tag{13}$$

where

$$\Phi(t,x,y,\tau) = [U(t,\tau) - 1/q] R(x) R(y),$$

$$U(t,\tau) = (2-q) e^{-(2T-t-\tau)q}/2q - e^{-|t-\tau|q}/2,$$

$$P(t,x,y) = [U(t,0)/(\gamma+q) - 1/\gamma q] R(x) R(y), \qquad \varepsilon(t,\tau) = \varepsilon(t,0) = 1,$$

$$G(t,x) = \int_0^1 P(t,x,y) h(0,y) \, dy + \frac{1}{q} \int_0^1 (e^{-(T-t)q} - 1) R(x) R(y) \psi(y) \, dy$$
$$- \int_t^T \int_0^1 e^{-(\tau-t)q} R(x) R(y) h(y,\tau) \, dy \, d\tau + h(t,x) + h(0,x)/\gamma + \psi(x).$$

Problem 2 becomes the problem of finding an optimal pair $\langle f, \varphi \rangle$ that realizes the best (in a certain sense) approximation of the process $u(t,x)$ to the prescribed process $h_1(t,x)$ over a certain time interval $(0,T)$. Its solution $f(x,t)$ is found from (12), (13), where

$$\Phi(t,x,\tau,y) = [Q(t,\tau) - \omega(t,\tau)/q]R(x)R(y), \qquad \omega(t,\tau) = \begin{cases} T-t, & 0 \leqslant \tau \leqslant t, \\ T-\tau, & t \leqslant \tau \leqslant T, \end{cases}$$

$$Q(t,\tau) = \left[(1-q^2)e^{-|t-\tau|q} - (1+q^2)e^{-(2T-t-\tau)q}\right]/2q^2, \qquad \varepsilon(t,\tau) = \omega(t,\tau),$$

$$P(t,x,y) = [Q(t,0)/(\gamma+q) - \omega(t,0)/\gamma q]R(x)R(y),$$

$$G(t,x) = \int_t^T \int_0^1 \left[\frac{e^{-(\tau-t)q} - 1}{q} h_1(\tau,y) - e^{-(\tau-t)q} h_2(\tau,y)\right] R(x)R(y)\,dy\,d\tau +$$

$$+ \int_t^T h_1(\tau,x)\,d\tau + \int_0^1 P(t,x,y)h_2(0,y)\,dy + h_2(t,x) + \frac{\omega(t,0)}{\gamma} h_2(0,x).$$

Other problems of this type, concerning problems of best approximation to a prescribed temperature regime, are contained in (4–7).

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Note: Figure translations are in progress. See original paper for figures.

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