

# ON PROPERTIES OF BOUNDED SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS IN $(B)$ -SPACES

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**Abstract**

**Full Text**

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*MATHEMATICS*

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## ON PROPERTIES OF BOUNDED SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS IN $B$ -SPACES

*(Presented by Academician V. I. Smirnov on 27 III 1967)*

1. Let  $Z$  be a complex  $B$ -space, and let  $A$  be some endomorphism\* of it with spectrum not intersecting the imaginary axis. We shall say that a set  $\Omega$ , whose elements are functions defined on  $(-\infty, +\infty)$  with values in  $Z$ , has property  $C$  if the equation

$$\dot{x} + Ax = \varphi(t) \quad (1)$$

has in  $\Omega$  a unique solution for every  $\varphi(t) \in \Omega$ .\*\*

It is known<sup>(1)</sup> that the property  $C$  is possessed by the set of functions continuous and bounded on  $(-\infty, +\infty)$ , the set of continuous periodic functions, and the set of almost periodic functions. It seems important to indicate a criterion allowing one to judge whether a given set  $\Omega$  possesses property  $C$ .

In the present paper this problem is solved for one family of sets  $\Omega$ : conditions are indicated that are necessary and sufficient in order that a set  $\Omega$  from this family possess property  $C$ . Some sufficient conditions are also indicated. The results of the monograph<sup>(1)</sup> listed above follow from the theorems of this paper.

2. We introduce notation.

$\Phi$  is the set of all strongly measurable and almost everywhere bounded functions with values in  $Z$ , defined on  $(-\infty, +\infty)$ .

$\nu$  is a functional defined on  $\Phi$  (the value  $+\infty$  is also allowed among its values) such that for any  $\varphi$  and  $\psi$  from  $\Phi$  and any complex  $\xi$  the following hold: 1)  $\nu(\varphi) \geq 0$ ; 2)  $\nu(\varphi + \psi) \leq \nu(\varphi) + \nu(\psi)$ ; 3)  $\nu(\xi\varphi) = |\xi|\nu(\varphi)$ ; 4)\*\*  $\nu[\varphi(t + \tau)] = \nu[\varphi(t)]$  for any  $\tau \in (-\infty, +\infty)$ ; 5) from the inequality  $\|\varphi(t)\| \leq \|\psi(t)\|$ , holding on  $(-\infty, +\infty)$ , follows the inequality  $\nu(\varphi) \leq \nu(\psi)$ .

$\Phi_\nu$  is the subset of  $\Phi$  formed by all elements of  $\Phi$  satisfying the inequality  $\nu(\varphi) < +\infty$ . It is obvious that  $\Phi_\nu$  is a complex linear system.

We shall call  $\varphi$  and  $\psi$  from  $\Phi$   $\nu$ -equivalent if  $\nu(\varphi - \psi) = 0$ . It is obvious that  $\Phi_\nu$  can be represented as a sum of nonintersecting classes of elements pairwise  $\nu$ -equivalent; the functional  $\nu$  is constant on each such class.

$C_\nu$  is the normed linear space whose elements are the above-mentioned classes of elements of  $\Phi_\nu$ , and the norm of an element is taken to be the value of the functional  $\nu$  on such a class. It is not difficult to see that all spaces  $L_p(-\infty, +\infty)$ ,  $1 \leq p \leq +\infty$ , are spaces of type  $C_\nu$ .

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\* By an endomorphism of a normed linear space this paper means a linear bounded operator mapping this space into itself.

\*\* A solution of the differential equation (1) is understood to be a continuous function satisfying this equation almost everywhere on  $(-\infty, +\infty)$ .

\*\*\* In connection with the use of the shift operation, the paper allows a double notation: sometimes we write  $\varphi \in \Phi$ , and sometimes  $\varphi(t) \in \Phi$ .

$E(X)$  is the set of endomorphisms of the linear normed space  $X$ . It is obvious that  $E(Z) \subset E(C_\nu)$ .

$P_\tau$  is the shift operator, i.e., the operator defined on  $\Phi$  by the equality  $P_\tau[\varphi(t)] = \varphi(t + \tau)$ ;  $P$  is the set of all  $P_\tau$ ,  $\tau \in (-\infty, +\infty)$ .

$S_h$  is the operator defined on  $\Phi$  by the equality\*

$$S_h[\varphi(t)] = \frac{1}{h} \int_t^{t+h} \varphi(\tau) d\tau;$$

$S$  is the set of all  $S_h$ ,  $h \neq 0$ ,  $h \in (-\infty, +\infty)$ . It is easy to see that  $P \subset E(C_\nu)$  and  $S \subset E(C_\nu)$ .

3. For the problem studied in this paper, the space  $C_\nu$  turns out to be a rather natural object, as is seen from the following theorem.

**Theorem 1.** *If  $\varphi(t) \in C_\nu$ , then equation (1) has a unique solution in  $C_\nu$ .*

In Theorem 1, the spaces  $L_p(-\infty, +\infty)$ ,  $1 \leq p \leq +\infty$ , may be taken as  $C_\nu$ . For  $p = +\infty$  one obtains the previously known result on the existence of a unique bounded solution of equation (1).

Despite its generality, Theorem 1 gives only one-sided information about the bounded solution of equation (1). To study more subtle properties of this solution, a new type of space is introduced.

**Definition.** Let  $\tilde{\Phi}$  be a complete subspace of the space  $C_\nu$ . We shall call  $\tilde{\Phi}$  a  $B$ -space if  $E(Z) \subset E(\tilde{\Phi})$ .

**Theorem 2.** \*Let  $\tilde{\Phi}$  be a  $B_\nu$ -space. In order that, for every  $\varphi(t) \in \tilde{\Phi}$  and every  $A^{**}$ , equation (1) have a unique solution in  $\tilde{\Phi}$ , it is necessary and sufficient that  $S \subset E(\tilde{\Phi})$ .\*

**Example.** Let  $\tilde{\Phi}$  be the set of all continuous bounded functions defined on  $(-\infty, +\infty)$  and having the limit  $\lim_{t \rightarrow +\infty} \varphi(t)$ . If  $\nu$  is introduced by the equality

$$\nu(\varphi) = \sup_{-\infty < t < +\infty} \|\varphi(t)\|,$$

then  $\tilde{\Phi}$  will be a  $B_\nu$ -space. For every  $\varphi(t) \in \tilde{\Phi}$  and  $h \neq 0$ , the function

$$\tilde{\varphi}(h, t) = \frac{1}{h} \int_t^{t+h} \varphi(\tau) d\tau$$

will be continuous; the limit will also exist:

$$\lim_{t \rightarrow +\infty} \tilde{\varphi}(h, t) = \lim_{t \rightarrow +\infty} \varphi(t).$$

Consequently,  $\tilde{\varphi}(t, h) \in \tilde{\Phi}$ , and the space  $\tilde{\Phi}$  introduced by us satisfies all the conditions of Theorem 2.

The conditions of Theorem 2 are checked just as simply for a broad class of spaces. In particular, these conditions are satisfied by the  $B_\nu$ -spaces of  $T$ -periodic functions

$$\left( \nu(\varphi) = \lim_{M \rightarrow \infty} \left( \frac{1}{2M} \int_{-M}^{+M} \|\varphi(t)\|^p \right)^{1/p}, 1 \leq p \leq +\infty \right);$$

uniformly almost periodic functions

$$\left( \nu(\varphi) = \sup_{-\infty \leq t < +\infty} \|\varphi(t)\| \right);$$

continuous functions, for each of which there exist two limits

$$\lim_{t \rightarrow -\infty} \varphi(t) \quad \text{and} \quad \lim_{t \rightarrow +\infty} \varphi(t)$$

$$\left( \nu(\varphi) = \sup_{-\infty < t < +\infty} \|\varphi(t)\| \right),$$

and many others.

Despite the convenience of using Theorem 2 in many concrete cases, it seems expedient to indicate some sufficient conditions that would imply the fulfillment of the condition  $S \subset E(\tilde{\Phi})$ , but not contain

\* Here and below the integral is understood in the sense of Bochner.

\*\* In accordance with the notation introduced earlier, this should be understood as follows:  $A$  is any element of the set of all endomorphisms of  $Z$  whose spectra do not intersect the imaginary axis.

would allow passage to the limit in the sense of the topology of the space  $Z$ . To obtain such results, however, one has to narrow the class of spaces under consideration.

Let us introduce the notation used in Theorems 3 and 3'.

Let  $\tilde{\Phi}$  be a  $B_\nu$ -space. We shall call  $\tilde{\Phi}$  a  $\tilde{B}_\nu$ -space if the following conditions are satisfied: a) every function that differs from zero only on a set of measure 0 is  $\nu$ -equivalent to 0; b) convergence of a sequence  $\{\varphi_n(t)\}$  in the space  $\tilde{\Phi}$  implies its convergence in measure on  $(-\infty, +\infty)$ .

Let  $\tilde{\Phi}$  be such a  $B_\nu$ -space that  $P \subset E(\tilde{\Phi})$ . We shall denote by  $\tilde{\Phi}_\tau = \{\varphi_\tau\}$  the set of all functions of the argument  $\tau$ , with values in  $\tilde{\Phi}$ , of the form  $\varphi_\tau = \varphi(t + \tau)$ , where  $\varphi(t) \in \tilde{\Phi}$ .

**Theorem 3.** *Let  $\tilde{\Phi}$  be a  $\tilde{B}_\nu$ -space such that  $P \subset E(\tilde{\Phi})$ . If there exists an  $h > 0$  such that all elements of the set  $\tilde{\Phi}_\tau$  are strongly measurable on  $[-h, +h]$ , then  $S \subset E(\tilde{\Phi})$ .*

Theorem 3 contains a requirement of measurability of functions from  $\tilde{\Phi}_\tau$  that is difficult to verify; more convenient for applications is its corollary,

**Theorem 3'.** *If  $\tilde{\Phi}$  is such a  $\tilde{B}_\nu$ -space that  $P \subset E(\tilde{\Phi})$  and every function  $\varphi_\tau$  from  $\tilde{\Phi}_\tau$  is continuous at the point  $\tau = 0$ , then  $S \subset E(\tilde{\Phi})$ .*

We note that there exist  $\tilde{B}_\nu$ -spaces satisfying the conditions of Theorem 2 but not satisfying the conditions of Theorems 3 and 3'. Such a space, in particular, is the space  $\tilde{\Phi}$  from the example to Theorem 2; the  $B_\nu$ -spaces of periodic and almost periodic functions also satisfy the conditions of Theorem 3'.

In conclusion, we note that the requirement of boundedness of the operator  $A$  can be weakened; it is required only that it belong to a certain class of closed operators.

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## REFERENCES

1. M. G. Krein, *Lectures on the Theory of Stability of Solutions of Differential Equations in Banach Space*, Kiev, 1964.

*Note: Figure translations are in progress. See original paper for figures.*

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