ON A CERTAIN CLASS OF EXTREMAL QUASICONFORMAL MAPPINGS

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ON A CERTAIN CLASS OF EXTREMAL QUASICONFORMAL MAPPINGS

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In this note a new class of extremal quasiconformal mappings is considered, connected with one generalization of the well-known Teichmüller theorem ([1–4]).

Let S and S' be two identically oriented closed Riemann surfaces of genus $g > 1$, and let a be a given homotopy class of orientation-preserving homeomorphisms $f : S \to S'$. On the surface S, let there be distinguished $n \geq 1$ distinct finitely connected domains $D_1, \dots, D_n$ with nondegenerate Jordan boundaries, with $\overline{D_i} \cap \overline{D_j} = \phi$ for $i \neq j$. Put

$$D = \bigcup_{j=1}^{n} D_j.$$

It is assumed that the surfaces S and S' are represented in the form

$$S = U/\Gamma, \qquad S' = U'/\Gamma',$$

where U and U' are respectively the disks $|z| < 1$ and $|w| < 1$, and Γ and Γ' are Fuchsian groups of the first kind corresponding to the surfaces S and S'. The groups Γ and Γ' are henceforth regarded as fixed.

We consider the following problems, which generalize Teichmüller' s problem.

Problem I. Let a divisor $\Delta = q_1^{\alpha_1} q_2^{\alpha_2} \dots q_l^{\alpha_l}$ be given on the surface S, where $q_i \in D$ and $\alpha_i \geq 0$ are integers. Let $P \subset U$ be a fixed fundamental polygon corresponding to the surface S, and let $z_1, \dots, z_l$ be points of P corresponding to the points $q_1, \dots, q_l$. Denote by $Q(a, \Delta, D)$, for some $Q < \infty$, the class of quasiconformal homeomorphisms $f : S \to S'$ with the following properties: 1) $f \in a$; 2) the homeomorphisms f are conformal in the domains D_j, $j = 1, \dots, n$; 3) the maximal dilatation* $K[f] \leq Q$; 4) the homeomorphisms f take prescribed values at the points z_i, together with their derivatives up to order α_i:

$$f^{(m)}(z_i) = w_{m,i}, \qquad m = 0, 1, \dots, \alpha_i; \quad i = 1, 2, \dots, l. \tag{1}$$

Let the class $Q(a, \Delta, D)$ be nonempty. It is required to find a mapping $f \in Q(a, \Delta, D)$ for which $\inf K[f]$ is attained in this class.

Problem II. It is required to find a mapping with the least dilatation $K[f]$ in the class of homeomorphisms $f : S \to S'$ satisfying conditions 1)–3) and the condition

$$f^{(\alpha_i)}(z_i) = w_{\alpha_i,i}; \qquad i = 1, 2, \dots, l. \tag{1'}$$

The solution of these problems is given by the following theorem.

Theorem. *Let S and S' be two closed Riemann surfaces of genus $g > 1$. Then, if the class $Q(a, \Delta, D)$ of quasiconformal homeomorphisms $f : S \to S'$ is nonempty, each of its extremal mappings $w = f_0(z)$ has the following properties: either $f_0(z)$ is an analytic function, or there exists a quadratic differential $\psi(w)\, dw^2 \not\equiv 0$ on S', having, possibly, at the points $q'_j = f_0(q_j)$ poles of order not higher than*

*

By the maximal dilatation of a quasiconformal mapping $w = f(z)$ with complex characteristic $\mu(z) = w_{\bar{z}}/w_z$ is meant

$$K[f] = (1 + k)/(1 - k),$$

where $k = \|\mu\|_{L_\infty(S)}$. In other words, $K[f]$ is the essential maximum of the characteristic $p(z)$ in the sense of M. A. Lavrent' ev ([5]).

$\alpha_j + 1$ $(j = 1, \dots, l)$ and holomorphic at the remaining points of the surface S', and such a positive constant $k < 1$ that the characteristic $\mu_0(w) = (f_0^{-1})_{\bar{w}}/(f_0^{-1})_w$ of the inverse mapping $z = f_0^{-1}(w)$ has the form

$$\mu_0(w) = \begin{cases} 0, & w \in D' = f_0(D), \\ k\overline{\psi(w)}/|\psi(w)|, & w \in S' \setminus D'. \end{cases} \tag{2}$$

The quadratic differential $\psi\, dw^2$ is determined uniquely up to a positive factor.

The extremal functions of Problem II possess analogous properties.

The existence of extremal mappings follows from the compactness of the classes under consideration. Let $w = f_0(z)$ be an extremal mapping in the class $Q(\alpha, \Delta, D)$. Put $q'_j = f_0(q_j)$, $D'_i = f_0(D_i)$, $D' = \bigcup_{i=1}^n D'_i$, $\Delta' = q_1'^{\alpha_1} \cdots q_{l'}^{\alpha_l}$, and denote by $A_1(U', \Gamma', \Delta', D')$ the Banach space of quadratic differentials $\psi\, dw^2$ on the surface S', having at the points q'_j poles of orders not exceeding $\alpha_j + 1$ and holomorphic at the remaining points of S', with norm

$$\|\psi\|_{A_1(U', \Gamma', \Delta', D')} = \iint_{S' \setminus D'} |\psi(w)|\, du\, dv < \infty \quad (w = u + iv). \tag{3}$$

By the Riemann–Roch theorem, $\dim A_1(U', \Gamma', \Delta', D') < \infty$. By $B(\Gamma')$ denote the Banach space of Beltrami differentials $\mu(w)\,d\bar{w}/dw$ on S' with norm $\|\mu\|_{B(\Gamma')} = \|\mu\|_{L_\infty(S')}$, and by $N(\Gamma', \Delta', D')$ the set of differentials $\nu(w)\,d\bar{w}/dw \in B(\Gamma')$ satisfying the conditions:

$$\nu(w) = 0, \quad w \in D'; \qquad \langle \nu, \psi \rangle = \iint_{S' \setminus D'} \nu(w)\psi(w)\,du\,dv = 0,$$

$$\psi \in A_1(U', \Gamma', \Delta', D'). \tag{4}$$

The proof of the theorem is carried out by the method of the work ([4]) and is based on the following lemmas.

Lemma 1. Let E_0 be a set of positive (Lebesgue) measure on the surface $S' = U'/\Gamma'$, with $E_0 \cap D' = \emptyset$, and let $\mu(w)d\bar{w}/dw$ be a Beltrami differential on S', equal to zero for $w \in E_0$ and $w \in D'$. Then there exists a differential $\hat{\mu}(w)d\bar{w}/dw \in N(\Gamma', \Delta', D')$, coinciding with $\mu(w)d\bar{w}/dw$ on $S' \setminus E_0$, and such that

$$\|\hat{\mu}\|_{L_\infty(E_0)} \leq C\|\mu\|_{L_\infty(S')}, \tag{5}$$

where C is a constant depending only on S', Δ', D', E_0.

Fix on the surface $S' = U'/\Gamma'$ a canonical dissection c, unique up to a homeomorphism of S' onto itself homotopic to the identity. This dissection determines in the fundamental group $\pi_1(S')$ of the surface S', up to an inner automorphism, a system of generators Σ': $a_1, b_1, \dots, a_g, b_g$, satisfying the relation

$$\prod_{j=1}^{g} a_{jb_{ja}} j^{-1} b_j^{-1} = 1.$$

Denote by $\tau_1, \tau_2, \dots, \tau_{3g-3}$ the moduli (coordinates), introduced by L. Ahlfors ([2]), of the pair (S', Σ'), i.e., the coordinates of the point $\bar{S}'$ determined by this pair in the Teichmüller space $T_g(S)$. The pair (S', Σ') is called a marked Riemann surface. If S^* is another closed surface of genus g, then every homeomorphism $f : S' \to S^*$ determines the corresponding marked surface (S^*, Σ^*), $\Sigma^* = f(\Sigma')$ (see ([2])). Fix fundamental polygons P' and P^*, corresponding to the marked surfaces (S', Σ') and (S^*, Σ^*).

Lemma 2. Let the marked Riemann surfaces (S', Σ') and (S^*, Σ^*) have respectively the coordinates $\{\tau_j'\}$ and $\{\tau_j^*\}$, $j = 1, \dots, 3g-3$, satisf⋯

satisfying the inequalities

$$|\tau_j' - \tau_j^*| < \delta, \quad j = 1, \dots, 3g-3. \tag{6}$$

Let a system of numbers $\Omega = \{\omega_{mi}\}$, $m = 0, 1, \dots, \alpha_i$, $i = 1, \dots, l$, be given, satisfying the conditions: $\omega_{0i} \in P^*$ and $|\omega_{0i} - w_i| < \delta$, where w_i are points of P' corresponding to the points $q_i' \in D'$; $|\omega_{1i} - 1| < \delta$, $|\omega_{mi}| < \delta$, $m = 2, 3, \dots, \alpha_i$; $i = 1, \dots, l$.

Then, for sufficiently small $\delta > 0$, there exists a Beltrami differential $\nu(w)\, d\overline{w}/dw$ such that $\nu(w) = 0$ for $w \in D'$, and the quasiconformal homeomorphism $\omega = f_\delta(w)$ with characteristic $\nu(w)$ maps the surface (S', Σ') onto (S', Σ^*) and has the following properties:

$$f^{(m)}(w_i) = \omega_{mi} \quad (m = 0, 1, \dots, \alpha_i;\ i = 1, \dots, l); \quad \|\nu\|_{L_\infty(S')} = O(\delta). \tag{7}$$

In particular, if the surfaces S' and S^* coincide, then we obtain a quasiconformal homeomorphism of the surface S' onto itself satisfying the conditions (7).

Lemma 3. Let the quasiconformal mapping $w = f_0(z) : S \to S'$ be extremal in the class $Q(a, \Delta, D)$, and let $f_{0\overline{z}}/f_{0z} = \nu_0(z)$. Then almost everywhere in $S \setminus D$ the inequality

$$|\nu_0(z)| = k_0 = (K_0 - 1)/(K_0 + 1), \quad K_0 = \inf_{f \in Q(a, \Delta, D)} K[f]. \tag{8}$$

holds.

Lemma 4. Let the quasiconformal mapping $w = f_0(z) : S \to S'$ be extremal in the class $Q(a, \Delta, D)$ and different from a conformal one on the set $S \setminus D$. Then there exists a quadratic differential $\psi_0(w)\, dw^2 \in A_1(U', \Gamma', \Delta', D')$ such that the inverse mapping $z = f_0^{-1}(w)$ has, at the points $w \in S' \setminus D'$, $D' = f_0(D)$, the characteristic

$$\mu_0(w) = k_0 \overline{\psi_0(w)}/|\psi_0(w)|. \tag{2'}$$

The differential $\psi_0\, dw^2$ is determined up to a positive constant factor.

Lemmas 1, 3, and 4 are proved analogously to the corresponding lemmas of the work [4]. The proof of Lemma 2 is based on the solution of a certain moment problem.

Along with closed surfaces, one may also consider compact Riemann surfaces bounded by a finite number of analytic curves, and require that the homeomorphisms under consideration also take, at a finite number of prescribed points $p_1, \dots, p_r$ lying in $S \setminus D$, prescribed values $p_1', \dots, p_r'$. However, this more general case is reduced in a known way to the one considered above. In this case the assertion of the theorem is preserved, only the quadratic differential $\psi\, dw^2$ must take real values on the boundary of the surface and may also have simple poles at the points $p_1', \dots, p_r'$.

Remark 1. Analogous theorems hold for mappings of the complex plane and of compact surfaces of genus $g = 1$ (tori).

We note the following assertion, analogous to Lemma 2 for $l = 1$.

Lemma 5. Let $d_0, d_1, \dots, d_n$ be prescribed complex numbers satisfying the inequalities $|d_0| < \varepsilon$, $|d_1 - 1| < \varepsilon$, $|d_s| < \varepsilon$, $s = 2, \dots, n$, and let $b_1, \dots, b_m$ be arbitrary finite points in the w-plane, distinct from $w = 0$. Then, for every $R > \max(|b_1|, \dots, |b_m|)$, for sufficiently small $\varepsilon > 0$, there exists a quasiconformal mapping $\omega = f_\varepsilon(w)$ of the w-plane onto itself having the following properties: 1) the mapping $f_\varepsilon(w)$ is conformal in the disk $|w| < R$; 2) $f_\varepsilon^{(s)}(0) = d_s$, $s = 0, 1, \dots, n$; 3) $f_\varepsilon(b_j) = b_j$, $j = 1, \dots, m$; 4) the characteristic $\mu_\varepsilon(w) = f'_{\varepsilon\overline{w}}/f'_{\varepsilon w}$ for $|w| > R$ satisfies the inequality $|\mu_\varepsilon(w)| < C(R)\varepsilon$, where $C(R)$ is a constant depending only on R.

Indeed, for $|w| \leq R$ one may put

$$\omega = f_\varepsilon(w) = \sum_{k=0}^{n+m} d_k w^k,$$

where the numbers $d_{n+1}, \dots, d_{n+m}$ are uniquely determined from the conditions $f_\varepsilon(b_j) = b'_j$, $j = 1, \dots, m$. By the conditions of the lemma we obtain that $f'_\varepsilon(w) = 1 + O(\varepsilon)$ for $|w| \leq R$, and $f_\varepsilon(w)$ is univalent in this disk for small ε. Then it is not difficult to construct a quasiconformal continuation of the mapping $f_\varepsilon(w)$ to the whole plane, in such a way that the characteristic $\mu_\varepsilon(w)$ for $|w| > R$ satisfies the inequality $|\mu_\varepsilon(w)| = 1 + O(\varepsilon)$.

Remark 2. One may also consider quasiconformal homeomorphisms $f : S \to S'$ with the following property. Let E be a set of positive measure on the surface S, with $m(S \setminus E) > 0$. It is required that the characteristic $\mu(z) = f_{\bar{z}}/f_z$ satisfy on the set E the inequality $|\mu(z)| \leq q(z)$, where $q(z)$ is a prescribed measurable nonnegative function on E such that $\|q\|_{L_\infty(E)} < 1$. If the class of such homeomorphisms is nonempty, then, analogously to the preceding, it is established that for the characteristics $\mu_0(z)$ of the extremal mappings $w = f_0(z)$, minimizing the quantity $\|\mu\|_{L_\infty(S \setminus E)}$, the equalities

$$|\mu_0(z)| = \begin{cases} q(z), & z \in E, \\ \text{const} < 1, & z \in S \setminus E. \end{cases} \tag{9}$$

hold. Such mappings for the case of rectangles were considered in ([6]).

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Note: Figure translations are in progress. See original paper for figures.