SovietRxiv

Soviet-era science, translated into English

ON THE EXISTENCE OF THE BROWN-McCOY RADICAL IN LIE ALGEBRAS

MATHEMATICS

1968

SovietRxiv

View the original and related papers at https://sovietrxiv.org/items/ru-196801.10582

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

SovietRxiv

Abstract

Full Text

UDC 519.48

MATHEMATICS

Academician of the Academy of Sciences of the MSSR V. A. ANDRUNAKIEVICH, Yu. M. RYABUKHIN

ON THE EXISTENCE OF THE BROWN-McCOY RADICAL IN LIE ALGEBRAS

In the present note the following is proved.

Theorem 1. *The only modular class in the class of all Lie algebras is the empty class of algebras.*

This theorem gives a negative answer to the question posed by Sulinski in the paper ([1]).

In order to formulate the question and prove the main theorem, let us recall some definitions. Everywhere below, unless the contrary is stated, by an algebra we mean a Lie algebra over some (fixed) associative-commutative ring Φ with identity 1. In particular, all Lie rings are allowed, if Φ coincides with the ring Z of integers. Let us recall that (see ([2]), p. 11) an algebra A over Φ is called a Lie algebra if $\forall x, y, z \in A$, $x^2 = 0$, $(xy)z + (yz)x + (zx)y = 0$. An algebra A is called simple if $A \neq 0$ and the only ideals in A are 0 and A.

If P is an ideal of an algebra A, then P is called a **retract** of the algebra A if there exists a homomorphism $\varphi : A \to P$ such that φ acts identically on P (see ([1]), Definition 4.2).

Let A be an arbitrary (not necessarily Lie) algebra over Φ. A linear mapping $D : A \to A$ is called a **derivation** of the algebra A if $\forall x, y \in A$,

$$(x, y)D = (xD)y + x(yD).$$

If A is a Lie algebra, then for any $a \in A$ the mapping $ada : x \to xa$ turns out to be a derivation. The derivations ada of the Lie algebra A are called **inner**. A Lie algebra A is called **perfect** if all its derivations are inner and, moreover, $\{a \in A \mid \forall x \in A,\ xa = 0\} = 0$.

With every Lie algebra over Φ one associates the holomorph $\mathfrak{H}(A)$ of the algebra A (see ([2]), p. 27)

$$\mathfrak{H}(A) = \{a + D \mid a \in A,\ D \text{ is a derivation in } A\},$$

$$(a_1 + D_1) + (a_2 + D_2) = (a_1 + a_2) + (D_1 + D_2),$$

SovietRxiv

$$(a_1 + D_1)(a_2 + D_2) = (a_1a_2 + a_1D_2 - a_2D_1) + (D_1D_2 - D_2D_1).$$

In this case the holomorph $\mathfrak{H}(A)$ of the Lie algebra A is itself a Lie algebra.

A class M of simple Lie algebras is called **modular** (see ([1]), Definitions 4.4 and 4.5) if the following requirements are satisfied:

I. If $P \in M$, P is an ideal of an algebra A, then P is a retract of the algebra A and there exists exactly one ideal T of the algebra A such that $A/T \in M$, $T \cap P = 0$.

II. If T is an ideal of an algebra B, B is an ideal of an algebra A, and $B/T \in M$, then T is an ideal of the algebra A.

In the paper ([1]) Sulinski shows that the class of all simple perfect Lie algebras satisfies requirement I and asks: does the class of all simple perfect Lie algebras satisfy requirement II, i.e., is this class modular?

Let us recall that if we are given some class M of algebras, then by S_M one denotes the property of being such an algebra A that no nonzero homomorphic image $\bar{A}$ of the algebra A belongs to the class M. S_M is called–

is called an upper radical defined by the class M, if the class M satisfies the requirement:

$(*)$ Every nonzero ideal of an algebra in the class M is homomorphically mapped onto nonzero algebras from M.

It is clear that every modular class satisfies $(*)$, since all algebras of this class are simple. The upper radical S_M, defined by the modular class M, Sulinskii calls the Brown–McCoy radical. This name is justified by the fact that, as shown in ([1]), in associative algebras modular classes are precisely the classes of simple algebras with identity, and the Brown–McCoy radical in associative algebras is the upper radical S_M defined by the class M of all simple algebras with identity.

We note that the empty class of algebras is, of course, modular. The empty class corresponds to the trivial radical: all algebras are radical. Therefore the question indicated by Sulinskii is equivalent to the following question: **does there exist a nontrivial Brown–McCoy radical in Lie algebras?** This also explains the title of the note.

Let us recall that, as is well known, simple perfect Lie algebras exist, for example, over a field of characteristic 0 (see below).

Proof of Theorem 1. We shall carry out the proof in stages, by contradiction. Suppose that a nonempty modular class of Lie algebras over Φ exists. Let this class be M.

1) Let A_X be the free associative algebra over Φ with the set X of free generators. We construct the Lie algebra A_X^L of the algebra A_X (see ([2]), p. 14), i.e. we pass to the left multiplication $[x, y] = xy - yx$.

SovietRxiv

Let $A_X^{(-)}$ be the Lie subalgebra of the Lie algebra A_X^L generated in A_X^L by the set X. By the results of A. I. Shirshov ([3]), $A_X^{(-)} = L_X$ is a free Lie algebra over Φ with the set X of free generators. This means that for any Lie algebra R over Φ and any mapping $\theta : X \to R$ there exists, and moreover exactly one, homomorphism $\varphi : L_X \to R$ coinciding with θ on X.

We shall assume that the set X is well ordered, $X = \{X_\alpha\}$. It is easy to see that for any $x_\alpha, x_\beta \in X$ there exists, and moreover exactly one, derivation $D_{\alpha\beta}$ of the algebra A_X such that $x_\alpha D_{\alpha\beta} = x_\beta$ and $x_\gamma D_{\gamma\beta} = 0$, if $\gamma \neq \alpha$. It is clear that $D_{\alpha\beta}$ is a derivation of the Lie algebra A_X^L. Indeed, if D is a derivation of the algebra A_X, then D is a derivation of the algebra $A_X^L = A_X$, since for any $x, y \in A_X = A_X^L$

$$[x, y]D = (xy - yx)D = (xy)D - (yx)D =$$

$$= (xD)y + x(yD) - (yD)x - y(xD) = [xD, y] + [x, yD].$$

Since $x \in A_X^{(-)} = L_X$, it is easy to see that the restriction of the mapping $D_{\alpha\beta}$ to $A_X^{(-)}$ (this mapping does not take $A_X^{(-)}$ out of $A_X^{(-)}$, i.e. $A_X^{(-)} D_{\alpha\beta} \subseteq A_X^{(-)}$) is a derivation of the algebra L_X. We shall denote this derivation by the same symbol $D_{\alpha\beta}$.

2) We now take into account that the class M is nonempty. Let $P \in M$. We well order the set P and construct the free Lie algebra L_P over Φ with the set P of free generators. We extend the identity mapping $\varepsilon : p \to P$ to a homomorphism $\varphi : L_P \to P$. It is clear that φ is a homomorphism onto P. Therefore

$$P \approx L_P \big/ \operatorname{Ker} \varphi. \tag{1}$$

We note that if 0 is the zero of the algebra P, then $0 = x_{\alpha_0}$ for some x_{α_0}, if $P = \{x_\alpha\}$. In this case $\varepsilon(x_{\alpha_0}) = 0$, and therefore also $\varphi(x_{\alpha_0}) = 0$. This means that $x_{\alpha_0} \in \operatorname{Ker} \varphi$.

But L_P is an ideal in $\mathfrak{H}(L_P)$, where $\mathfrak{H}(L_P)$ is the holomorph of the Lie algebra L_P. Since $P \in M$ and $\operatorname{Ker} \varphi$ is an ideal in L_P, it follows from (1) and requirement II that

$\operatorname{Ker} \varphi$ is an ideal in $\mathfrak{H}(L_P)$. This means that $(\operatorname{Ker} \varphi)D \subseteq \operatorname{Ker} \varphi$ for every derivation D of the algebra L_P. In particular,

$$\forall x_\beta \in P x_{\alpha_0} \in \operatorname{Ker} \varphi \Longrightarrow x_{\alpha_0} D_{\alpha_0\beta} \in \operatorname{Ker} \varphi.$$

SovietRxiv

Therefore $P \subseteq \operatorname{Ker}\varphi$. But P is the set of generators of the algebra L_P. Consequently, $L_P = \operatorname{Ker}\varphi$ and therefore, by (1), $P = 0$. Meanwhile $P \neq 0$—a contradiction. The theorem is proved.

From the theorem just proved there follows a negative answer both to the question posed by Sulinski and to the question posed by us (since it is equivalent to Sulinski' s question): the class of all simple perfect Lie algebras is not modular if it is nonempty, and there exist no nontrivial Brown—McCoy radicals.

Let us now note that the negative answer was obtained by making essential use of infinite-dimensional algebras. In this connection it is natural to consider also the finite-dimensional case. We shall restrict ourselves to the case where Φ is a field.

Thus, everywhere below we assume that all Lie algebras under consideration are finite-dimensional Lie algebras over the field Φ. Note that in this case the characteristic p of the field Φ plays an essential role.

It is clear that if A is a finite-dimensional Lie algebra, then its holomorph $\mathfrak{H}(A)$ is also a finite-dimensional Lie algebra.

Theorem 2. *If the characteristic $p > 0$, then the only modular class in finite-dimensional Lie algebras is the empty class.*

Proof. Suppose the contrary. Let M be a modular class and $R \in M$. Consider the associative-commutative algebra A over Φ with basis $\{1, z, z^2, \dots, z^{p-1}\}$ and defining relation $z^p = 0$. Note that the mapping $D : A \to A$, defined by the rule $1D = 0$, $z^{iD} = iz^{i-1}$, $1 \leq i < p$, extends to a derivation of the algebra A.

Construct the tensor product $B = R \otimes A$ of the algebra R by the algebra A. It is clear that $R \otimes A$ is a finite-dimensional Lie algebra (see ([2]), p. 40, Exercise 23, and p. 88, example).

Extend the derivation D of the algebra A to a derivation of the algebra B, putting

$$\left(\sum r_i \otimes a_i\right) D = \sum r_i \otimes a_{iD}.$$

Now note that $R \otimes zA$ is an ideal in $R \otimes A$, and $R \otimes A \mid R \otimes zA \approx R \otimes \Phi \approx R \in M$. Therefore $R \otimes zA$ is an ideal also in the holomorph $\mathfrak{H}(B)$ of the algebra $B = R \otimes A$, by II. In particular, $(R \otimes zA)D \subseteq R \otimes zA$, and hence $(R \otimes zA)D + R \otimes zA = R \otimes zA$. Meanwhile $(R \otimes zA) + (R \otimes zA)D \supseteq R \otimes zA + R \otimes \Phi \supseteq R \otimes A \supset R \otimes zA$, since $R \neq 0$—a contradiction. The theorem is proved.

Let us now recall that a Lie algebra A is called **abelian** if $xy = 0$ for all $x, y \in A$.

Theorem 3. *If the characteristic $p = 0$, then any class of simple nonabelian algebras is modular. Every modular class consists of simple nonabelian algebras.*

Proof. The only simple abelian Lie algebra over Φ is the one-dimensional algebra $\Phi x = \{\alpha x \mid \alpha \in \Phi,\ x^2 = 0,\ x \neq 0\}$. Consider the two-dimensional

SovietRxiv

Lie algebra $\Phi x + \Phi y$, in which $xy = x$ (see ([2]), p. 19, example). Φx is an ideal in $\Phi x + \Phi y$. Moreover, if $\varphi : R \to P$ is a homomorphism of the algebra $R = \Phi x + \Phi y$ onto the algebra $P = \Phi x$ and φ acts identically on P, then $x = \varphi(x) = \varphi(xy) = \varphi(x)\varphi(y) = 0$, and we obtain a contradiction. Therefore a modular class cannot contain abelian algebras.

Now take any class M of simple nonabelian algebras. It is clear that every simple nonabelian Lie algebra will be semisimple in the usual "classical" sense (see ([2]), p. 34). Therefore every simple nonabe-

the left algebra of the Lie algebra is perfect (see (2), p. 87). But then, as shown by Sulinsky (see (2), p. 19, Proposition 3), the class M satisfies condition I.

It remains to prove that condition II is fulfilled (i.e., to answer Sulinsky' s question in the affirmative in the case under consideration!). This is easily done using Levi' s theorem (see (2), p. 105).

Indeed, let T be an ideal in B, B an ideal in A, and suppose $B/T \in M$. Then, as is easily seen, in view of what was proved above, the radical $R = \mathrm{rad}(B)$ of the algebra B is contained in T. By Levi' s theorem, there exists a semisimple subalgebra C in B such that $B = P + C$, $P \cap C = 0$. We note that for any $a \in A$, $Ra \subseteq R$, since ada is a derivation of the algebra B (see (2), p. 64). Therefore ada induces a derivation r_a of the algebra C, defined by the following rule: if $c \in C$, then ($ca \in B$!).

$$cr_a = c',$$

where $ca = c' + b'$ for some $b' \in P$. But C is a perfect algebra (see (2), p. 87), and therefore there exists an element $c_a \in C$ such that r_a coincides with adc_a on C. Then, for any $t \in T$, representing t in the form $t = b + c$, $b \in R$, $c \in C$, and for any $a \in A$, we obtain, for some $b' \in R$,

$$ta = ba + ca = ba + cr_a + b' = ba + cadc_a + b' = ba + cc_a + b' \in T,$$

since $c_a \in B$, $c = t - b \in T$, T is an ideal in B, $b' \in R$, $ba \in R$. Consequently, T is an ideal in A. The theorem is proved.

It is easy to show that, in the case of finite-dimensional Lie algebras of characteristic zero, the Brown-McCoy radical coincides with the classical solvable radical.

Received
20 XI 1967

CITED LITERATURE

[1] A. Sulinsky, *Fund. math.*, **59**, No. 1, 23 (1966). [2] N. Jacobson, *Lie Algebras*, Moscow, 1965. [3] A. I. Shirshov, *Mat. sbornik*, **45**(87), 2, 113 (1958).

SovietRxiv

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.