# ON A GENERAL QUADRATURE PROCESS AND ITS APPLICATION TO THE APPROXIMATE SOLUTION OF SINGULAR INTEGRAL EQUATIONS

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## Abstract

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*MATHEMATICS*

**B. G. GABDULKHAEV**

# ON A GENERAL QUADRATURE PROCESS AND ITS APPLICATION TO THE APPROXIMATE SOLUTION OF SINGULAR INTEGRAL EQUATIONS

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The subject of the present note is the investigation of one general quadrature process for singular integrals with a Hilbert-type kernel and its application to the approximate solution of the corresponding singular integral equations by the method of mechanical quadratures.

**1°.** Let $\tilde{x} = \tilde{x}(s)$ be a trigonometric polynomial of degree $n$ interpolating a complex-valued $2\pi$-periodic function $x = x(s)$ at the pairwise distinct roots of the equation $\sin(ns + \omega) = 0$, where $\omega$ is an arbitrary real number. It is known ([1]) that $\tilde{x}$ is determined up to the term $\delta \sin(ns + \omega)$, where $\delta$ is an arbitrary constant. Determining $\delta$ from the requirement that the functional

$$\varphi(\tilde{x}) = \|\tilde{x}\|^2_{\mathcal{L}2} = \frac{1}{\pi} \int_0^{2\pi} |\tilde{x}(s)|^2 \, ds$$

be minimal, we obtain the interpolation formula

$$P_n x = \frac{1}{2n} \sum_{k=0}^{2n-1} x(s_k) \sin n(s - s_k) \operatorname{ctg} \frac{s - s_k}{2}, \tag{1}$$

which is exact for any trigonometric polynomial of degree not exceeding $n - 1$.

**2°.** Consider the singular integral ([2]) with a Hilbert-type kernel

$$Ix = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{ctg} \frac{\sigma - s}{2} \, x(\sigma) \, d\sigma, \tag{2}$$

which is understood in the sense of the Cauchy–Lebesgue principal value. Approximating the density $x(s)$ by the trigonometric interpolation polynomial of the special form (1), we find for the integral (2) the following quadrature formulas:

$$Ix = \frac{1}{n} \sum_{k=0}^{2n-1} x_k \sin^2 n \frac{s_k - s}{2} \operatorname{ctg} \frac{s_k - s}{2} + R_n, \tag{3}$$

$$(Ix)_i = \frac{1}{n} \sum_{k=0}^{2n-1}{}' x_k \operatorname{ctg} \frac{s_k - s_i}{2} + R_{ni} \quad (i = 0, 1, \dots, 2n-1), \tag{4}$$

where $f_i = f(s_i)$, $R_n = R_n(s) = R_n(x; s)$ is the remainder term, and $\sum'$ means that summation is carried out only over those $k$ for which the difference $k - i$ is odd. It is clear that the trigonometric degree of precision of these formulas is equal to $n - 1$.

We note that the method under consideration for the approximate evaluation of singular integrals is very effective. In particular, this method makes it possible to obtain, in a unified way, a number of known quadrature formulas for the integral (2), previously found from other considerations.

For example, for $\omega = 0$ and even $n$, formula (4) yields the corresponding formula of [3], and for $\omega = 0$, the formula of [4].

In estimating the error of the quadrature process (3)–(4), $R_n$ will be regarded as a linear operator from one linear normed space $X$ into another space $Y$. Choosing $X$ and $Y$ in an appropriate way, we obtain the following assertion.

**Theorem 1.** *The quadrature process (3)–(4) converges:*

a) *in the mean for any density $x(s)$ integrable in the Riemann sense; if $x(s)$ is a continuous function, then for $n = 1, 2, \dots$ the estimate*

$$\|R_n(x; s)\|_{\mathfrak{L}} \le (1 + \sqrt{2}) E_{n-1}(x) \qquad (x \in C), \tag{5}$$

*is valid, where $E_n(x)$ is the best uniform approximation of $x(s)$ by trigonometric polynomials of degree $n$;*

b) *uniformly for any density satisfying the Dini–Lipschitz condition; if $x(s)$ is continuous in the Hölder sense with any positive exponent $\alpha$, then for $n = 2, 3, \dots$*

$$\|R_n(x; s)\|_C \le (A \ln n + B) E_{n-1}(x) \qquad (x \in H_\alpha), \tag{6}$$

*where $A$ and $B$ are quite definite constants independent of $x$ and $n$.*

We note that Theorem 1, in combination with the known estimates of N. I. Akhiezer–M. G. Krein–J. Favard (see [5]) for best approximations, makes it possible to obtain effective uniform and mean-square estimates of the error of formulas (3)–(4) for densities from certain classes of functions.

3°. Consider the singular integral equation [2] with a Hilbert-type kernel

$$ax + bIx + Mx = y, \tag{7}$$

where $z = Mx$,

$$z(s) = \frac{1}{2\pi} \int_0^{2\pi} m(s,\sigma) x(\sigma)\, d\sigma,$$

and the $2\pi$-periodic functions $a = a(s)$, $b = b(s)$, $m = m(s,\sigma)$ (in both arguments), and $y = y(s)$ have $r$ continuous derivatives and belong to the class $H_\alpha$ $(r \geq 0,\ 0 < \alpha \leq 1)$.

As an approximate solution of equation (7) we take a polynomial of the form (1):

$$x_n(s) = \frac{1}{2n} \sum_{k=0}^{2n-1} \alpha_k \sin n(s - s_k) \operatorname{ctg} \frac{s - s_k}{2}. \tag{8}$$

Using formulas (1) and (4), to determine the coefficients $\{\alpha_k\}$ we find the following system of linear algebraic equations:

$$a_i \alpha_i + \frac{b_i}{n} \sum_{k=0}^{2n-1} \operatorname{ctg} \frac{s_k - s_i}{2} \alpha_k + \frac{1}{2n} \sum_{k=0}^{2n-1} m_{ik} \alpha_k = y_i, \tag{9}$$

where $m_{ik} = m(s_i, s_k)$, $\psi_i = \psi(s_i)$ $(i = 0, \dots, 2n-1)$.

With regard to the computational scheme (7)–(9), the following is valid.

**Theorem 2.** *Suppose that $a^2 + b^2 \neq 0$ anywhere on $[0, 2\pi]$ and equation (7) has a unique solution $x(s)$ for any right-hand side. Then, for $n$ satisfying the inequality*

$$q = \frac{L \ln n + K}{n^{r+\alpha}} < 1 \quad (L = \text{const},\ K = \text{const}), \tag{10}$$

*system (9) is uniquely solvable and the approximate solution (8) converges uniformly to the exact solution with rate*

$$|x(s) - x_n(s)| \leq \frac{R \ln n + Q}{n^{r+\alpha}} \quad (R = \text{const},\ Q = \text{const}). \tag{11}$$

We note that, for the proof of the theorem, equation (7) is reduced to the particular form of one generalized Riemann boundary-value problem (see [2]), and also

(^6)) and prove the stability of the solutions of this problem under perturbations in the uniform metric, in particular when its coefficients are replaced by polynomials of the form (1). Next, with the aid of the lemma of paper (^6), we justify the method of mechanical quadratures for the "perturbed" equation (here one has essentially to use the results on approximation by polynomials (1) in the metric of the space $W$, introduced by V. V. Ivanov ([7]·8)). To complete the proof we use the device proposed in paper (⌒6).

4°. Let us note that for a singular integral equation of the form

$$a(s)x(s) + \frac{1}{2\pi} \int_0^{2\pi} m(s,\sigma) \operatorname{ctg} \frac{\sigma - s}{2}\, x(\sigma)\, d\sigma = y(s) \tag{12}$$

the corresponding algebraic system has the form

$$a_i \alpha_i + \frac{1}{n} \sum_{k=0}^{2n-1}{}' m_{ik} \operatorname{ctg} \frac{s_k - s_i}{2}\, \alpha_k = y_i \quad (i = 0, \dots, 2n-1). \tag{13}$$

Thus, for equation (12), the system of the method of mechanical quadratures considered above coincides with the system of Multhopp' s method (⌒9). Hence, and from the results of V. V. Ivanov ([7]·8), it follows that here, in contrast to scheme (7)–(9), in the general case an assertion analogous to Theorem 2 cannot be proved.

5°. From what was said in items 3° and 4° it follows that the computational scheme (7)–(9) has a significant advantage over the scheme (8), (12), (13). In practice, system (9) can be solved by Gaussian elimination with pivoting, as well as by other methods. In the case of a characteristic equation ($m \equiv 0$), the computation is somewhat simplified and can be carried out, in particular, by Multhopp' s method of successive approximations (⌒9).

An analysis of numerical examples computed on the M-20 and Ural-2 machines, and of the theoretical justification carried out (Theorems 1 and 2), shows the high efficiency of the computational schemes (7)–(9) of the method of mechanical quadratures. Moreover, the method is the more effective, the better the structural properties of the coefficients of the equation being solved. In particular, if the functions $a$, $b$, $m$ (in both arguments) and $y$ have bounded derivatives of arbitrary order, then one may assert that, under the conditions of Theorem 2, system (9) has a unique solution for all $n \geq 2$, and for these $n$ the interpolation polynomial (8) provides a very effective approximation to the exact solution of equation (7).

6°. In conclusion, we note that the results given above can also be transferred to singular integrals with a Cauchy-type kernel and to the corresponding integral equations specified along sufficiently smooth closed contours. For example, in the case of a circle such a transfer is especially simple and can be carried out with the help of relations (2), which connect integral (2) and the corresponding singular integral with a Cauchy-type kernel.

Kazan State University
named after V. I. Ulyanov (Lenin)

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*Note: Figure translations are in progress. See original paper for figures.*