ON THE DIMENSION OF PRODUCTS OF BICOMPACTA

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Abstract

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ON THE DIMENSION OF PRODUCTS OF BICOMPACTA

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§ 1. In the present section only finite covers and only normal spaces are considered. By the dimension of a space X we mean the large inductive dimension, denoted by $\operatorname{Ind} X$. Let us recall the definition of this dimension: $\operatorname{Ind} X \leq n$ if, for every closed set $F \subseteq X$ and every neighborhood OF of it, there exists a neighborhood V of the set F such that $V \subseteq OF$ and $\operatorname{Ind}([V] \setminus V) \leq n-1$. By definition $\operatorname{Ind} X = -1$ if the space X is empty.

Definition 1. For a space X, a closed set F_1 is called a **boundary** if F_1 is the boundary of a set open in X.

Definition 2. A space X satisfies the condition of extension of refinements*, if for any closed sets F_1 and F_2, $F_1 \supseteq F_2$, $\operatorname{Ind} F_1 > \operatorname{Ind} F_2$, F_2 a boundary in F_1, the following holds: for every open cover ω of the set F_1 and every refinement α of the set F_2, inscribed in the cover $\omega \wedge F_2$** with $\operatorname{Ind} \operatorname{gr} \alpha \leq \operatorname{Ind} F_2 - 1$, there exists a refinement β of the set F_1, inscribed in the cover ω, such that $\beta \wedge F_2 = \alpha$, $\operatorname{gr} \beta \cap F_2 = \operatorname{gr} \alpha$, and $\operatorname{Ind} \operatorname{gr} \beta \leq \operatorname{Ind} F_2 - 1$.

Definition 3. We shall say that a bicompactum X **belongs to the class of bicompacta** if the following conditions are fulfilled for X:

I (weakened sum theorem). For any two closed sets F_1 and F_2 that are boundaries,

$$\operatorname{Ind}(F_1 \cup F_2) = \max(\operatorname{Ind} F_1, \operatorname{Ind} F_2).$$

II. The bicompactum X satisfies the condition of extension of refinements.

It turns out that for bicompacta X belonging to the class , the following is true.

Theorem 1. Let

$$P = \prod_{i=1}^{s} X_i,$$

where $X_i \in$ $(i = 1, 2, \dots, s)$. Then

$$\operatorname{Ind} P \leq \operatorname{Ind} X_1 + \dots + \operatorname{Ind} X_s.$$

It is not hard to show that the class of bicompacta includes the following classes of bicompacta: a) all bicompacta whose large inductive dimension does not exceed one; b) all bicompacta that are Dowker spaces[***] (see (2)), and consequently also all perfectly normal bicompacta.

The proof of part a) is contained in (4).

Corollary 1. Let $P = X \times Y$, where X is a one-dimensional bicompactum and Y is a perfectly normal bicompactum for which $\dim Y = \operatorname{ind} Y = \operatorname{Ind} Y = n$. Then $\dim P = \operatorname{ind} P = \operatorname{Ind} P = n + 1$.

Proof. $\dim P \geq n+1$ follows from (5), and $\operatorname{Ind} P \leq n+1$ follows from Theorem 1.

The proof of Theorem 1 proceeds as follows:

1) It is shown that in any open covering ω of the bicompactum P one can inscribe a refinement α^1 such that $\operatorname{bd} \alpha^1$ is the sum of a finite number of summands of the form $\prod_{i=1}^{s} F_i$, where F_i is the boundary of some order in X_i, and

$$\sum_{i=1}^{s} \operatorname{Ind} F_i = \sum_{i=1}^{s} \operatorname{Ind} X_i - 1$$

for each summand of the set $\operatorname{bd} \alpha^1$.

2) Then it is shown that in any open covering ω^1 of the set $\operatorname{bd} \alpha^1$ one can inscribe a refinement α^2 of the set $\operatorname{bd} \alpha^1$ such that $\operatorname{bd} \alpha^2$ (relative to $\operatorname{bd} \alpha^1$)

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[*] A cover $\alpha = \{[O_i]\}$ $(i = 1, \dots, n)$ is called a **refinement** if its elements are the closures of pairwise disjoint open sets O_i, called the kernels of the elements of the refinement,

$$\operatorname{gr} \alpha = \bigcup_{j=1}^{n} [O_i] \setminus O_i.$$

[**] Let $F \subseteq X$: 1) let $\omega = \{u_1, \dots, u_s\}$ be an open cover of the space X; then $\omega \wedge F = \{F \cap u_1, \dots, F \cap u_s\}$; 2) let $\alpha = \{[O_1], \dots, [O_s]\}$ be a refinement of X; then $\alpha \wedge F = \{[O_1 \cap F], \dots, [O_n \cap F]\}$.

[***] A hereditarily normal space X is called a **Dowker space** if every open set U is covered by a point-finite system of open sets F_σ in X.

is the sum of a finite number of closed summands of the form $\prod_{i=1}^{s} F_i'$, and, for each summand of the set $\operatorname{bd}\alpha^2$, we have

$$\sum_{i=1}^{s} \operatorname{Ind} F_i' = \sum_{i=1}^{s} \operatorname{Ind} X_i - 2 \quad (F_i' \subseteq X_i').$$

After this procedure has been applied $\sum_{i=1}^{s} \operatorname{Ind} X_i = l$ times, we obtain that $\operatorname{bd}\alpha^l$ is the sum of a finite number of closed summands of the form $\prod_{i=1}^{s} F_i$, and

$$\sum_{i=1}^{s} \operatorname{Ind} F_i = 0.$$

Consequently, the dimension of each set F_i is zero; therefore the dimension of each summand of the set $\operatorname{bd}\alpha^l$ and the dimension of the set $\operatorname{bd}\alpha^l$ itself is zero. Then, going back, we obtain that

$$\operatorname{Ind} \operatorname{bd}\alpha \leq \sum_{i=1}^{s} \operatorname{Ind} X_i - 1$$

and, consequently,

$$\operatorname{Ind} P \leq \sum_{i=1}^{s} \operatorname{Ind} X_i.$$

Definition 4. We shall say that a bicompactum X **belongs to the class of bicompacta** A if X is a closed subset of some bicompactum P which is the topological product of a finite number of bicompacta from the class .

Remark 1. The class A is closed with respect to topological product.

Remark 2. Let $B = \prod_{i=1}^{k} A_i$, where $A_i \in A$ $(i = 1, \dots, k)$ and each bicompactum A_i is finite-dimensional. Then the bicompactum B is also finite-dimensional.

§ 2. In this section we consider the question of the dimension of the product of an infinite number of arbitrary bicompacta X with $\dim X \geq 1$.

The following notion of a closed partition between disjoint closed sets is known. Let C_1 and C_2 be closed sets of a space X, $C_1 \cap C_2 = \emptyset$. Then a closed set B is called a **partition** between C_1 and C_2 if $X \setminus B = V^1 \cup V^2$, $V^1 \cap V^2 = \emptyset$, the sets V^1, V^2 are open, and $C_1 \subseteq V^1$, $C_2 \subseteq V^2$.

P. S. Aleksandrov gave the following definition: a space X is **strongly infinite-dimensional** if there exists a sequence of pairs (C_i^1, C_i^2), $C_i^1 \cap C_i^2 = \emptyset$ $(i =$

$1, 2, \ldots)$, of closed sets such that the intersection of any closed sets B_i $(i = 1, 2, \ldots)$, each of which is a partition for the corresponding pair (C_i^1, C_i^2), is nonempty. If the space X is strongly infinite-dimensional in the sense of P. S. Aleksandrov, then we shall say that it is A-strongly infinite-dimensional.

Now we introduce a strengthening of the notion of partition.

Definition 5. Let there be given closed sets C^1 and C^2 in X, with $C^1 \cap C^2 = \emptyset$. A set B is called a **thick partition** between C^1 and C^2 if $X \setminus \operatorname{Int} B = \Phi_1 \cup \Phi_2$, $\Phi_1 \cap \Phi_2 = \emptyset$, the sets Φ_1, Φ_2 are closed, $C^i \subseteq \operatorname{Int} \Phi_i$ $(i = 1, 2)$ ($\operatorname{Int} B$ denotes the interior of the set B), and $B \cap (C^1 \cup C^2) = \emptyset$.

If the space X is normal, then the following assertions are equivalent:

I. In the space X there exist n pairs of closed sets (C_i^1, C_i^2) $(i = 1, 2, \ldots, n)$, $C_i^1 \cap C_i^2 = \emptyset$, such that the intersection of any closed sets B_i $(i = 1, 2, \ldots, n)$, each of which is a thick partition for the pair (C_i^1, C_i^2), is nonempty.

II. In the space X there exist n pairs of closed sets (C_i^1, C_i^2) $(i = 1, 2, \ldots, n)$, $C_i^1 \cap C_i^2 = \emptyset$, such that the intersection of arbitrary closed sets B_i $(i = 1, 2, \ldots, n)$, each of which is a partition for the pair (C_i^1, C_i^2), is nonempty.

Definition 6. A normal space X will be called **L-strongly infinite-dimensional** if there exists a sequence of pairs (C_i^1, C_i^2) $(i = 1, 2, \ldots)$, $C_i^1 \cap C_i^2 = \emptyset$, of closed sets such that the intersection of arbitrary closed sets B_i $(i = 1, 2, \ldots)$, each of which is a thick partition for the corresponding pair (C_i^1, C_i^2), is nonempty.

Lemma 1. *If the space X is a bicompactum, then the definitions of L-strong infinite-dimensionality and A-strong infinite-dimensionality are equivalent.*

Proof. a) From A-strong infinite-dimensionality, obviously, L-strong infinite-dimensionality follows.

b) Let the bicompactum X be L-strongly infinite-dimensional. Suppose that it is not A-strongly infinite-dimensional. Then for any sequence of pairs of closed sets (F_i^1, F_i^2), $F_i^1 \cap F_i^2 = \emptyset$ $(i = 1, 2, \ldots)$, there exists a sequence of closed sets A_i $(i = 1, 2, \ldots)$ (each of which is a partition respectively for the pair (F_i^1, F_i^2)) such that

$$\bigcap_{i=1}^{\infty} A_i = \emptyset.$$

Consequently, also for the sequence of pairs of closed sets (C_i^1, C_i^2) participating in the definition of L-strong infinite-dimensionality, there exists a corresponding sequence of closed sets A_i $(i = 1, 2, \ldots)$. Since X is a bicompactum and

$$\bigcap_{i=1}^{\infty} A_i = \emptyset,$$

there exists an integer k such that

$$\bigcap_{i=1}^{k} A_i = \emptyset.$$

Therefore one can take open sets O_i $(i = 1, 2, \dots, k)$ such that $A_i \subseteq O_i$, $[O_i] \cap (C_i^1 \cup C_i^2) = \emptyset$, and

$$\bigcap_{i=1}^{k} [O_i] = \emptyset.$$

Now, taking for each $j > k$ an arbitrary open set O_j such that $A_j \subseteq O_j$ and $[O_j] \cap (C_j^1 \cap C_j^2) = \emptyset$, we obtain such a sequence of closed sets $[O_i]$ $(i = 1, 2, \dots)$ (which are thick partitions for the corresponding pairs (C_i^1, C_i^2)) that

$$\bigcap_{i=1}^{\infty} [O_i] = \emptyset.$$

We have arrived at a contradiction with the choice of the sequence of pairs of closed sets (C_i^1, C_i^2) $(i = 1, 2, \dots)$.

Remark 3. Lemma 1 is also true for countably paracompact spaces (spaces in every countable open cover of which one can inscribe a locally finite cover (see $(^6)$)).

Theorem 2. *Let* $P = \prod_{\alpha \in A} X_\alpha$*, where* X_α *are bicompacta,* $\dim X_\alpha \geq 1$ *for every* $\alpha \in A$*, and* $m(A) \geq \aleph_0$*. Then the bicompactum* P *is L-strongly infinite-dimensional.*

Proof. It is enough to consider the case when

$$P = \prod_{i=1}^{\infty} X_i$$

is the topological product of a countable number of bicompacta X_i. Since $\dim X_i \geq 1$, in the bicompactum X_i there exist points $x_i^1 \neq x_i^2$ which cannot be separated by the empty set. Take the sets

$$C_i^1 = \prod_{j \neq i} X_j \times x_i^1, \qquad C_i^2 = \prod_{j \neq i} X_j \times x_i^2.$$

Then C_i^1 and C_i^2 are closed in the bicompactum P, $C_i^1 \cap C_i^2 = \emptyset$, and it remains only to prove that the intersection of arbitrary closed sets B_i $(i = 1, 2, \dots)$, each of which is a thick partition for the pair (C_i^1, C_i^2), respectively, is nonempty. To prove the last assertion, we note that for this it is sufficient to prove the nonemptiness of the intersection of any finite number of the sets B_i, since then the assertion we need will follow from the bicompactness of the space P. Thus, the proof of Theorem 2 has been reduced to the proof of the following assertion.

Let

$$P_n = \prod_{i=1}^{n} X_i,$$

where the X_i are bicompacts and $\dim X_i \geqslant 1$ for every $i = 1, 2, \dots, n$. Then the sequence of pairs of closed sets (C_i^1, C_i^2) (where

$$C_i^1 = \prod_{j \neq i}^{n} X_j \times x_i^1, \qquad C_i^2 = \prod_{j \neq i}^{n} X_j \times x_i^2$$

) is such that the intersection of arbitrary closed sets B_i $(i = 1, 2, \dots, n)$, each of which is a thick partition for the pair (C_i^1, C_i^2), is nonempty.

We shall prove the last assertion for the case $n = 2$. Let

$$P_2 = \prod_{i=1}^{2} X_i, \qquad \dim X_i \geqslant 1 \quad (i = 1, 2),$$

and let B_1, B_2 be thick partitions for the pairs of closed sets (C_1^1, C_1^2) and (C_2^1, C_2^2), respectively. Suppose that $B_1 \cap B_2 = \emptyset$. Then the sets

$$F_i^1 = B_i \cap \Phi_i^1, \qquad F_i^2 = B_i \cap \Phi_i^2$$

(where $\Phi_i^1 \cup \Phi_i^2 = P_2 \setminus \operatorname{Int} B_i$ $(i = 1, 2)$ and $\Phi_i^1 \cap \Phi_i^2 = \emptyset$) are pairwise disjoint, i.e.

$$F_i^1 \cap F_i^2 = \emptyset$$

for $i = 1, 2$. For fixed i, the sets $F_i^1, F_i^2, C_i^1, C_i^2$ are also pairwise disjoint. Therefore one can take an open cover ω of the bicompact P_2 such that:

$$(*)$$

no element of this cover intersects simultaneously an arbitrary pair of disjoint sets from the system

$$F_i^1,\ F_i^2,\ B_i,\ \Phi_i^1,\ \Phi_i^2,\ C_i^1,\ C_i^2 \quad (i = 1, 2).$$

We refine this cover ω by an open cover

$$\tilde{\omega} = \{\omega_1 \times \omega_2\},$$

whose elements are sets of the form $(u_i^1 \times u_j^2)$, where $u_i^1 \in \omega_1$, $u_j^2 \in \omega_2$, and ω_1, ω_2 are open covers of the bicompacts X_1, X_2, respectively. Let $\pi_i : P_2 \to X_i$ be the natural projection of P_2 onto the factor X_i. Then one may require that no element of the cover ω_i intersect a pair of sets $(x_i^1, \pi_i(B_i))$ and $(x_i^2, \pi_i(B_i))$. Now take, for each ω_i, an ω_i-map f into the body of the nerve $\widetilde{K}_i$ of the cover (see (2)). Then the points $f_i(x_i^1)$ and $f_i(x_i^2)$ lie in one component of the polyhedron $\widetilde{K}_i$, and therefore there exists a segment I_i whose initial point is $f_i(x_i^1)$ and whose endpoint is $f_i(x_i^2)$. Let

$$X_i^+ = f_i^{-1}(I_i).$$

The set X_i^+ is closed in X_i, and $x_i^j \in X_i^+$ $(j = 1, 2)$. The mapping

$$f = f_1 \times f_2$$

of the bicompact P_2 maps the set

$$P_2^+ = X_1^+ \times X_2^+$$

onto a square, and the sets

$$P_2^+ \cap C_i^j = +C_i^j = x_i^j \times X_k^+ \quad (i = 1, 2;\ i = k;\ k = 1, 2;\ j = 1, 2)$$

are mapped onto opposite sides of the square for fixed i. Let

$$\omega^+ = P_2^+ \wedge \omega;$$

then

$$f^+ = f/P_2^+,$$

the restriction of the mapping f to the set P_2^+, will be an ω^+-map, and

$$B_i^+ = B_i \cap P_2^+.$$

The set $f^+(B_i^+)$ separates the sets $f^+(+C_i^1)$ and $f^+(+C_i^2)$, since f^+ is an ω^+-map and the cover ω of the bicompact P_2 satisfies condition $(*)$. Consequently,

$$f^+(B_1^+) \cap f^+(B_2^+) = \emptyset,$$

and this contradicts the fact that

$$B_1^+ \cap B_2^+ = \emptyset,$$

f^+ is an ω^+-map, and no element of the cover ω^+ intersects simultaneously the sets B_1^+ and B_2^+. Consequently,

$$B_1^+ \cap B_2^+ = P_2^+ \cap (B_1 \cap B_2) \neq \emptyset,$$

i.e.

$$B_1 \cap B_2 \neq \emptyset.$$

Theorem 2 is proved.

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References

1. P. S. Alexandroff, Proc. Roy. Soc., A **189**, No. 2 (1947).
2. P. S. Alexandroff, *Combinatorial Topology*, 1947.
3. B. A. Pasynkov, DAN, **175**, No. 2 (1967).
4. I. K. Lifanov, DAN, **180**, No. 3 (1968).
5. H. Cohen, Duke Math. J., **21**, No. 2, 209 (1954).
6. C. H. Dowker, Canad. J. Math., **3**, No. 2, 219 (1951).

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