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Abstract

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MATHEMATICS

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FACTORIZATION OF MAPPINGS INTO METRIC SPACES

(Presented by Academician P. S. Aleksandrov on 29 I 1968)

All spaces under consideration are assumed to be normal. For a finite space X we set $w(X) = \aleph_0$. The first factorization theorem for a mapping of a bicomactum onto a bicomactum was obtained by S. Mardešić in ⁽¹⁾; the first factorization theorem for a mapping of a normal space into a metric one was obtained by the author in ⁽²⁻⁴⁾; for further results see ^(5,6). Mardešić's factorization theorems from ⁽¹⁾ and mine from ^(3,4) are definitive in the sense that in them factorization is carried out both with respect to dimension and with respect to weight. At the same time, in the papers ^(5,6) factorization is carried out mainly with respect to dimension (it is not clear whether one may assume, for example, $w(hA_\lambda) \leq w(fA_\lambda)$ in Theorem 2 of ⁽⁵⁾, or $w(hF_i) \leq w(fF_i)$ in Theorem 3 of ⁽⁶⁾*). It turns out that in the case of mappings into metric spaces complete factorization (both with respect to weight and with respect to dimension) is possible.

Theorem 1. a) *If the system π of closed sets A_α , $\alpha \in \mathfrak{A}$, of a space X is σ -locally finite, then for every mapping f of the space X into a metric space R there exist a metric space S and mappings $g : X \rightarrow S$ and $h : S \rightarrow R$ such that: 1) $f = hg$, and the mapping h is perfect; 2) $w(B) \leq w(hB)$ for every set $B \subset S$; 3) $\dim S \leq \dim X$, $\dim[gA_\alpha] \leq \dim A_\alpha$ for every $\alpha \in \mathfrak{A}$.*

- b) *If the system π of sets A_α , $\alpha \in \mathfrak{A}$, having type F_σ in X , is σ -strongly locally countable**, then for every mapping f of the space X into a metric space R there exist a metric space S and mappings $g : X \rightarrow S$ and $h : S \rightarrow R$ such that assertions 1) and 2) of part a) hold; in addition: 3) $\dim S \leq \dim X$, and for every $\alpha \in \mathfrak{A}$ the set gA_α is contained in a set B_α of type F_σ in S with $\dim B_\alpha \leq \dim A_\alpha$.*

If in part b) the space X is paracompact, then the condition on the system π can be weakened: it is enough to assume that it is σ -locally countable***.

Basic definitions. A system π of normal subsets A_α , $\alpha \in \mathfrak{A}$, of a space X will be called: a) **weakly factorizable**,

* It is likewise not clear whether $\dim[hF_i] \leq \dim F_i$.

** We shall call a system π of subsets of a space X strongly locally countable if there exists a locally finite open cover of the space X , each element of which intersects no more than a countable set of elements of the system π . A system that can be decomposed into the sum of a countable number of strongly locally countable subsystems will be called σ -strongly locally countable.

*** The formulated Theorem 1 is a more general and stronger assertion than Theorem 3 from the paper ⁽⁶⁾ of A. V. Arhangel'skii. Incidentally, we note that the derivation given by him in ⁽⁶⁾ of Theorem 6 from Theorem 3 of that paper is not convincing. For the derivation to be correct, in Theorem 3 of ⁽⁶⁾ it is necessary additionally to require the validity of the relations $\dim[hF_i] \leq \dim F_i$, which is formulated in Theorem 1 above, in particular, and is achieved. I note further that neither in ⁽²⁾ nor in ^(3,4) did I use spectral technique in proving the factorization theorem (see the remark in ⁽⁶⁾).

b) **factorizable**, c) **strongly factorizable**, d) **strongly* factorizable** with respect to a mapping $f : X \rightarrow Y$, if there exist a space Z and mappings $g : X \rightarrow Z$ and $h : Z \rightarrow Y$ such that the following relations hold: 1) $f = hg$; 2) $w(gA_\alpha) \leq w(fA_\alpha)$ for every $\alpha \in \mathfrak{A}$; 3) respectively: a) $\dim g\tilde{A}_\alpha^* \leq \dim A_\alpha$, $\alpha \in \mathfrak{A}$; b) $\dim[gA_\alpha]_Z \leq \dim A_\alpha$, $\alpha \in \mathfrak{A}$; c) the mapping h is perfect** and $\dim[gA_\alpha]_Z \leq \dim A_\alpha$, $\alpha \in \mathfrak{A}$; d) the mapping h is perfect and for every $\alpha \in \mathfrak{A}$ the set gA_α is contained in a set B_α of type F_σ in Z with $\dim B_\alpha \leq \dim A_\alpha$. If $w(B) \leq w(hB)$ for every set $B \subseteq Z$, then we shall say that the system π is **factorizable without narrowing**. If a system π of subsets of the space X is (strongly, strongly*, weakly) factorizable (without narrowing) with respect to every mapping of the space X into any space Y from the class of spaces \mathfrak{M} in such a way that $Z \in \mathfrak{M}$, then the system π will be called ****(strongly, strongly*, weakly) factorizable**** (without narrowing) with respect to the class of spaces \mathfrak{M} .

In what follows we shall be interested mainly in the factorization of mappings into metrizable spaces and into spaces which in a certain sense are close to them.

I. It turns out that the factorization of mappings into metrizable spaces reduces to the case of mappings into spaces with a countable base; namely, the following is true.

Reduction Lemma 1. *If a system π of subsets A_α , $\alpha \in \mathfrak{A}$, of a space X : a) is weakly factorizable, b) factorizable, c) strongly factorizable, or d) strongly* factorizable with respect to the class \mathfrak{M}_c of spaces with a countable base, then the system π is factorizable without narrowing in the corresponding sense (a), b), c), or d)) with respect to the class \mathfrak{M}_p of metric spaces.*

Definition. A property Q of a system of sets of a space X will be called **(strongly, strongly*, weakly) factorizable (without narrowing)**, respectively **(strongly, strongly*, weakly) τ -factorizable (without narrowing)**

with respect to a class of spaces \mathfrak{M} , if every system of sets of the space X possessing the property Q , respectively every system of sets of the space X decomposable into the sum of τ subsystems possessing the property Q , is (strongly, strongly*, weakly) factorizable (without narrowing) with respect to every mapping of the space X into a space from the class \mathfrak{M} . In the case $\tau = \aleph_0$ we shall call the property Q not \aleph_0 -factorizable, but σ -factorizable.

Lemma on σ . *If a property Q of a system of sets of a space X is (strongly, weakly) factorizable with respect to the class \mathfrak{M}_c of spaces with a countable base, then the property Q is (strongly, weakly) σ -factorizable with respect to the class of all spaces with a countable base and, consequently, with respect to the class of all metric spaces (in the latter case without narrowing).*

Definition. We shall say that a system π of normal subsets A_α , $\alpha \in \mathfrak{A}$, of a space X : a) is **dim-closed**; b) **has type F_σ with respect to the dimension dim**, if respectively: a) for the closure $[A_n]$ of the sum A_n of all sets A_α with $\dim A_\alpha \leq n$, the relation $\dim[A_n] \leq n$, $n = 0, 1, 2, \dots$, holds; b) the sum A_n of all sets A_α with $\dim A_\alpha \leq n$ has type F_σ in X and $\dim A_n \leq n$.

It is clear that a locally finite system of closed sets of a space X is dim-closed, and a σ -locally finite system of sets of type F_σ will have type F_σ with respect to the dimension dim. It is also clear that

* $g\tilde{A}_\alpha$ is some normal subset of the space Z containing gA_α .

** Consequently, the set $h(Z)$ is closed in Y .

a system π decomposing into a countable sum of systems having type F_σ with respect to the dimension dim, will itself be of the same kind. Thus, if the property of a system of sets of having type F_σ with respect to the dimension dim turns out to be factorizable in some sense with respect to a class of spaces \mathfrak{M} , then this property will also be σ -factorizable (in the corresponding sense) with respect to the class \mathfrak{M} .

Lemma. A σ -strongly locally countable system π of sets of type F_σ in a space X has type F_σ with respect to the dimension dim.

Theorem 1 follows from the following assertion.

Proposition 1. a) The property of a system π of subsets of a space X of being dim-closed is strongly σ -factorizable without enlargement with respect to the class of metric spaces. b) The property of a system π of sets of spaces X of having type F_σ with respect to the dimension dim is strongly* σ -factorizable without enlargement with respect to the class of metric spaces.

It is possible to factor not only one mapping into a metric space, but any countable family of such mappings:

Theorem 2. a) If a system π of subsets A_α , $\alpha \in \mathfrak{A}$, of a space X decomposes into a countable sum of dim-closed subsystems π_j , $j = 1, 2, \dots$, then for any countable collection of mappings f_i , $i = 1, 2, \dots$, of the space X into metric spaces

R_i , there exists a metric space S and mappings $g : X \rightarrow S$ and $h_i : S \rightarrow R_i$ such that: 1) $f_i = h_i \cdot g$, $i = 1, 2, \dots$; 2) $w(B) \leq \max_i(w(h_i B))$ for every set $B \subseteq S$, in particular $w(S) \leq \max_i(w(R_i))$; 3) $\dim S \leq \dim X$, $\dim[gA_\alpha] \leq \dim A_\alpha$; 4) the product $h : S \rightarrow \prod R_i$ of the mappings h_i is perfect; in particular, if all the spaces R_i are complete, then S will also be complete.

- b) If in part a) the systems π_j , $j = 1, 2, \dots$, are taken not to be dim-closed, but to have type F_σ with respect to the dimension \dim , then assertions 1), 2) and 4) remain valid, and instead of assertion 3) the following assertion is true: 3') $\dim S \leq \dim X$, and for every $\alpha \in \mathfrak{A}$ the set gA_α is contained in a set B_α of type F_σ in S with $\dim B_\alpha \leq \dim A_\alpha$.

From Theorem 2 and the theorem of § 1 of [7] it follows that

Theorem 3. 1) For any σ -locally finite (σ -locally countable) system π of closed (respectively, of type F_σ) subsets A_α , $\alpha \in \mathfrak{A}$, of a metrizable space X ; 2) for any countable system of homeomorphisms h_i , $i = 1, 2, \dots$, of the space X onto itself and 3) for any countable system of completions \tilde{X}_j , $j = 1, 2, \dots$, of the space X with respect to some metrics on X , there exists a metric on X such that for the completion \tilde{X} of the space X with respect to this metric the following holds: 1) $\dim \tilde{X} = \dim X$, $\dim A_\alpha = \dim[A_\alpha]_{\tilde{X}}$, $\alpha \in \mathfrak{A}$ (respectively, $\dim \tilde{X} = \dim X$, and for each $\alpha \in \mathfrak{A}$ the set A_α is contained in a set B_α of type F_σ in \tilde{X} with $\dim B_\alpha \leq \dim A_\alpha$); 2) the homeomorphisms h_i extend to homeomorphisms \tilde{h}_i of the completion \tilde{X} onto itself, $i = 1, 2, \dots$; 3) for every $j = 1, 2, \dots$ there exists a mapping of \tilde{X} into \tilde{X}_j which is the identity on X .

This theorem is, in a certain sense, a generalization of Theorem II.10 of [8].

Theorem 3 permits the following strengthening of the theorem on approximation of an n -dimensional space, complete in the sense of Dieudonné, by a spectrum of n -dimensional metric spaces.

Theorem 4. Let the space X be complete in the sense of Dieudonné, and let a system π of sets A_α , $\alpha \in \mathfrak{A}$, of type F_σ in X be σ -strongly locally countable (if X is paracompact, then the system π may be merely σ -locally countable). Then there exists a spectrum $S = \{R_\lambda, \mathfrak{F}_\lambda^\mu\}$; $\lambda \in \Lambda$, of metric spaces R_λ with projections “onto,” such that: 1) the space X is the limit of the spectrum S and $\dim R_\lambda \leq \dim X$, $\lambda \in \Lambda$; 2) for every $\alpha \in \mathfrak{A}$ the set A_α is the limit of the spectrum $S_\alpha = \{\mathfrak{F}_\lambda A_\alpha, \mathfrak{F}_\lambda^\mu\}$, $\lambda \in \Lambda$ (\mathfrak{F}_λ

denotes the projection of the limit X of the spectrum S onto the element R_λ of this spectrum) and $\dim \mathfrak{F}_\lambda A_\alpha \leq \dim A_\alpha$, $\lambda \in \Lambda$.

Remark. In fact, not only $\dim \mathfrak{F}_\lambda A_\alpha \leq \dim A_\alpha$, $\lambda \in \Lambda$, but in each space R_λ , for every α there exists a set B_α^λ of type F_σ in R_λ such that $B_\alpha^\lambda \supset \pi_\lambda A_\alpha$ and $\dim B_\alpha^\lambda \leq \dim A_\alpha$. If, moreover, the system π consists of closed sets and is σ -locally finite, then one may take $\dim[\pi_\lambda A_\alpha] \leq \dim A_\alpha$, $\lambda \in \Lambda$, $\alpha \in \mathfrak{A}$.

II. A zero-dimensional perfect mapping onto a metric space will be called

an **improvement**, and spaces possessing an improvement will be called **improvable**. The class of improvable spaces is in many respects similar to the class of metric spaces, as is shown, in particular, by the following assertion.

Reduction Lemma 2. *If the system π of subsets A_α , $\alpha \in \mathfrak{A}$, of the space X : a) is weakly factorizable, b) factorizable, c) strongly factorizable, or d) strongly* factorizable with respect to the class \mathfrak{M} of metric spaces, then the system π is factorizable without restriction in the corresponding sense (a), b), c), or d)) with respect to the class \mathfrak{M}_a of improvable spaces. In this case the sets $\tilde{g}A_\alpha$ in case a) and B_α in case d) (see the basic definitions) may be considered improvable spaces.*

From Reduction Lemma 2 it follows immediately that

Assertion 1. *Proposition 1 and Theorem 1 remain valid if in them the class of metric spaces is replaced by the class of improvable spaces.*

Since the product of a countable system of improvements is an improvement, it follows that

Assertion 2. *Theorem 3 remains valid if in it the spaces R_i , $i = 1, 2, \dots, S$, and the sets B_α are regarded as improvable.*

The analogue of Theorem 2 is also valid:

Assertion 3. *1) For any σ -locally finite (σ -locally countable) system π of closed (respectively, of type F_σ) subsets A_α , $\alpha \in \mathfrak{A}$, of an improvable space X , and 2) for any countable system of complete in the sense of Čech improvable extensions \tilde{X}_j , $j = 1, 2, \dots$, of an improvable space X , there exists a complete in the sense of Čech improvable extension \tilde{X} of the space X such that: 1) $\dim \tilde{X} = \dim X$, $\dim[A_\alpha]_{\tilde{X}} = \dim A_\alpha$, $\alpha \in \mathfrak{A}$ (respectively, for each α the set A_α is contained in a set B_α of type F_σ in \tilde{X} with $\dim B_\alpha \leq \dim A_\alpha$); 2) for any $j = 1, 2, \dots$ there exists a mapping of \tilde{X} into \tilde{X}_j , identical on X .*

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