ESTIMATES OF SOLUTIONS OF GENERAL BOUNDARY-VALUE PROBLEMS FOR PARABOLIC SYSTEMS IN \(C_{l+\alpha(r)}(Q)\)

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Abstract

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ESTIMATES OF SOLUTIONS OF GENERAL BOUNDARY-VALUE PROBLEMS FOR PARABOLIC SYSTEMS IN $C_{l+\alpha(r)}(Q)$

(Presented by Academician I. G. Petrovskii on 24 X 1967)

In (1) Schauder-type estimates were obtained for solutions of general boundary-value problems for general parabolic systems. For systems parabolic in the sense of I. G. Petrovskii, in (2) Schauder-type estimates were obtained inside the domain in $[\cdot]_{i+\alpha(r)}$ ($\alpha(r)$ is the refined Hölder exponent) under the condition that

$$Ar^{\alpha(r)} = \int_0^r t^{\alpha(t)-1}\,dt < \infty. \tag{*}$$

S. N. Kruzhkov in (3) showed that condition $(*)$ is minimal under which Schauder-type estimates are possible.

In the present paper we obtain, under the minimal condition $(*)$, estimates for solutions of general parabolic systems inside the domain and for solutions of the Cauchy problem. For solutions of general boundary-value problems the estimates are obtained under the condition that $\alpha(r) \in \mathcal{A}_l$, where by $\mathcal{A}_l$ we denote the class of refined Hölder exponents for which $A^l r^{\alpha(r)} < \infty$.

Consider in a domain $\mathcal{D}$ of the $(n+1)$-dimensional space $(x_1, x_2, \dots, x_n, t) = (x, t)$ the system of differential equations

$$\mathcal{L}(x, t; D_x, D_t)u(x, t) = f(x, t), \tag{1}$$

where $u = (u_1, u_2, \dots, u_m)$; $f = (f_1, f_2, \dots, f_m)$; $\mathcal{L}(x, t; D_x, D_t)$ is a square matrix with elements $l_{kj}(x, t; D_x, D_t)$.

System (1) is called parabolic if there exist integers s_k, t_j $(k, j = 1, 2, \dots, m)$ such that:

1) The degree of the polynomial $l_{kj}(a, t; i\xi\lambda, p\lambda^{2b})$ with respect to the variable λ at each point $(x, t) \in \mathcal{D}$ does not exceed $s_k + t_j$, and if $s_k + t_j < 0$, then $l_{kj} = 0$. Let $\mathcal{L}^0$ be the principal part of the matrix $\mathcal{L}$.
2) There exists a constant $\delta > 0$ such that the roots of the polynomial $L(x, t; i\xi, p) = \det \|\mathcal{L}^0(x, t; i\xi, p)\|$ with respect to the variable p, for any real ξ, satisfy the inequality

$$\operatorname{Re} p_s \leq -\delta|\xi|^{2b}$$

for all points $(x, t) \in \mathcal{D}$.

Let

$$\sum (s_i + t_i) = 2br, \qquad r > 0, \qquad \max_i s_i = 0.$$

By $C_{l+\alpha(r)}(\mathcal{D})$ we shall denote the space of functions $v(x, t)$ for which the norm is finite

$$|v|^D_{l+\alpha(r)} = \sum_{j=0}^{l} [v]^D_j + [v]^D_{l+\alpha(r)},$$

where

$$[v]^{\mathcal{D}}_j = \sum_{(2b\mu+\nu)=j} \sup_{\mathcal{D}} |D^\mu_t D^\nu_x v(x, t)|\,,$$

$$[v]^{\mathcal{D}}_{l+\alpha(r)} = [v]^{\mathcal{D}}_{l+\alpha(r),x} + [v]^{\mathcal{D}}_{l+\alpha(r),t} \equiv$$

$$\equiv \sum_{(2\mu+\nu)=l} \sup_{(x,t),(x',t)\in\mathcal{D}} \frac{|D^\mu_t D^\nu_x v(x, t) - D^\mu_t D^\nu_x v(x', t)|}{|x - x'|^{\alpha(|x-x'|)}} +$$

$$+ \sum_{0<l-2b\mu-\nu<2b} \sup_{(x,t),(x,t')\in\mathcal{D}} \frac{|D^\mu_t D^\nu_x v(x, t) - D^\mu_t D^\nu_x v(x, t')|}{|t - t'|^{(l-2b\mu-\nu+\alpha(|t-t'|^{1/2b}))/2b}}.$$

For $\alpha(r) \in \mathcal{A}_1$ set

$$B\alpha(r) = \frac{1}{\ln r} \ln \frac{Ar^{\alpha(r)}}{(Ar^{\alpha(r)})(1)}.$$

Theorem 1. Let $u(x, t) = (u_1(x, t), \ldots, u_m(x, t))$ be a solution of system (1) in a bounded domain $\mathcal{D}$, and let the derivative $D^\mu_t D^\nu_x u_j(x, t)$, $2b\mu + \nu = t_j$, be continuous in $\mathcal{D}$. Let the coefficients of the operators $l_{k_j}(x,t;D_{xD}t)$ be bounded by a constant K_1 in the norms $|\cdot|^{\mathcal{D}}_{l-s_k+\alpha(r)}$, $l \geq 0$, $\alpha(r) \in \mathcal{A}_1$. Let

$$M = \sum_i \left(|f_i|^{\mathcal{D}}_{l-s_i+\alpha(r)} + |u_i|^{\mathcal{D}}_0 \right) < \infty.$$

Then the estimate holds

$$|u_j|^{\mathcal{D}'}_{t_j+l+B\alpha(r)} \leq KM, \qquad j = 1, 2, \dots, m,$$

where $\overline{\mathcal{D}}' \subset \mathcal{D}$, and the constant K depends on $\mathcal{D}'$, K_1, n, m, δ, $s_1, \dots, s_m$, $t_1, \dots, t_m$, the diameter of the domain $\mathcal{D}$, and the function $\alpha(r)$.

Consider in $\mathcal{D}^{(T)}_{n+1} = E_n \times [0,T]$, $T < \infty$, the Cauchy problem for system (1)

$$\mathcal{L}(x,t;D_x,D_t)u(x,t) = f(x,t), \qquad \mathcal{E}(x;D_x,D_t)u(x,t)\big|_{t=0} = \varphi(x), \tag{2}$$

where $\mathcal{E}(x;D_x,D_t)$ is a matrix with elements $C_{\gamma j}(x,D_x,D_t)$ ($\gamma = 1,2,\dots,r$, $j = 1,2,\dots,m$). Suppose that there exist integers ρ_γ such that the degree of the polynomial $C_{\gamma j}(x, i\xi\lambda, p\lambda^{2b})$ in λ does not exceed $\rho_\gamma + t_j$, and if $\rho_\gamma + t_j < 0$, then $C_{\gamma j} = 0$. Let $\mathcal{E}$ satisfy the complementarity condition (see (1), § 1).

Theorem 2. Let the coefficients of the operators l_{kj}, $C_{\gamma j}$ be bounded by a constant K_2 in the norms $|\cdot|^{\mathcal{D}_{n+1}^{(T)}}_{l-s_k+\alpha(r)}$, $|\cdot|^{E_n}_{l-\rho_\gamma+\alpha(r)}$, respectively, $l \geq 0$, $\alpha(r) \in \mathcal{A}_1$. Then, for arbitrary

$$f_k(x,t) \in C_{l-s_k+\alpha(r)}(\mathcal{D}^{(T)}_{n+1}), \qquad \varphi_\gamma(x) \in C_{l-\rho_\gamma+\alpha(r)}(E_n),$$

the Cauchy problem (2) has a unique solution with

$$u_j(x,t) \in C_{l+t_j+B\alpha(r)}(\mathcal{D}^{(T)}_{n+1})$$

and

$$\sum_{j=1}^{m} |u_j|^{\mathcal{D}_{n+1}^{(T)}}_{l+t_j+B\alpha(r)} \leq K \left(\sum_{j=1}^{m} |f_j|^{\mathcal{D}_{n+1}^{(T)}}_{l-s_j+\alpha(r)} + \sum_{\gamma=1}^{r} |\varphi_j|^{E_n}_{l-\rho_\gamma+\alpha(r)} \right),$$

where the constant γ depends on δ, K_2, n, ρ_γ, s_i, t_j, $Ar^{\alpha(r)}$, and the constant in the complementarity condition.

Let $Q = \Omega \times [0,T]$, $T < \infty$, where Ω is a bounded domain with boundary $S \in C_{l+t_{\max}+\alpha(r)}$; $\Gamma = S \times [0,T]$.

Consider in Q the problem

$$\mathcal{L}u = \mathcal{L}(x, t; D_{xD}t)u(x, t) := f(x, t),$$

$$\mathcal{E}(x, D_{xD}t)u(x, t)\big|_{t=0} = \varphi(x), \tag{3}$$

$$\mathcal{B}(x, t; D_{xD}t)u(x, t)\big|_{\Gamma} = \Phi(x, t),$$

where $\mathfrak{B}(x, t; D_x, D_t)$ is a matrix with elements $B_{qj}(x, t; D_x, D_t)$ $(q = 1, \dots, br)$. Suppose that there exist integers σ_q such that the degree of the polynomial $B_{qj}(x, t; i\xi\lambda, p\lambda^{2b})$ in λ does not exceed $\sigma_q + t_j$. Suppose that the matrices $\mathfrak{B}$ and $\mathfrak{E}$ satisfy the complementarity conditions (see (^{1}), § 1).

Let $x \in S$ be an arbitrary point. Introduce a local coordinate system with center at the point x. Suppose that in this neighborhood the coefficients of the operators l_{kj} in the local coordinates do not depend on the normal coordinate for $0 \le t \le T$.

Theorem 3. Let the coefficients of the operators $l_{kj}, C_{\gamma j}, B_{qj}$ be bounded by the constant K_3 in the norms

$$|\cdot|^{Q}_{l-s_k+\alpha(r)}, \qquad |\cdot|^{\Omega}_{l-\rho_\gamma+\alpha(r)}, \qquad |\cdot|^{\Gamma}_{l-\sigma_q+B\alpha(r)}$$

respectively, $l \ge \sigma_0 = \max(0, \sigma_1, \dots, \sigma_{br})$, $\alpha(r) \in A_4$.

If $f_j \in C_{l-s_j+\alpha(r)}(Q)$, $\varphi_j \in C_{l-\rho_\gamma+\alpha(r)}(\Omega)$, $\Phi_q \in C_{l-\sigma_q+B\alpha(r)}(\Gamma)$ and the compatibility condition of order l is fulfilled (see (^{1}), § 14), then problem (3) has a unique solution

$$u(x, t) = (u_1(x, t),\ u_2(x, t), \dots, u_m(x, t))$$

with

$$u_j(x, t) \in C_{t_j+l+B^2\alpha(r)}(Q)$$

and

$$\sum_{j=1}^{m} |u_j|^{Q}_{l+t_j+B^2\alpha(r)} \le K \left(\sum_{j=1}^{m} |f_j|^{Q}_{l-s_j+B\alpha(r)} + \sum_{j=1}^{r} |\varphi_j|^{Q}_{l-\rho_\gamma+\alpha(r)} + \sum_{q=1}^{br} |\Phi_q|^{\Gamma}_{l-\sigma_q+B\alpha(r)} \right),$$

where the constant K depends on $\delta, K_3, n, s_i, t_j, \rho_\gamma, \sigma_q, Ar^{\alpha(r)}$ and the constants of the complementarity condition.

The proofs of these theorems are based on the results obtained in (^{1}) for problems with constant coefficients (2), (3), in which only the principal parts of the matrices $\mathfrak{L}, \mathfrak{B}, \mathfrak{C}$ enter, and on the following estimates for the basic potentials (for notation see (^{1}), § 11) for $l \geq 0$, $\alpha(r) \in A_1$:

$$[(\Gamma * f)]_{l+2br+B\alpha(r)}^{{E_{n+1}}^{(T)}} \leq C[f]_{l+\alpha(r)}^{{E_{n+1}}^{(T)}},$$

$$[(\Gamma *_1 \varphi)]_{l+2b(r-1)+B\alpha(r)}^{{\mathfrak{D}_{n+1}}^{(T)}} \leq C[\varphi]_{l+\alpha(r)}^{E_n},$$

$$[((K_{jq}^{Q} *_2 \Phi)]_{l+2br-s_j+\sigma_q+2bQ+B\alpha(r)}^{\widetilde{\mathfrak{D}}_{n+1}^{(T)}} \leq C|\Phi|_{l+\alpha(r)}^{\widetilde{E}_n^{(T)}}$$

for $2br - s_j + \sigma_q + 2bQ > 0$; $0 < t < T \leq \infty$, where $\mathfrak{D}_{n+1}$ is the half-space $x_n > 0$ in E_{n+1}, E_n is the space $x_n = 0$ in E_{n+1}.

Remark. Since in the proof only the fact is used that the function $r^{\alpha(r)}$ has the properties of a modulus of continuity and of a continuous derivative for $r > 0$, one may take as the refined Hölder exponent the function

$$\alpha(r) = \frac{1}{\ln r} \ln \frac{\omega(r)}{\omega(1)},$$

where $\omega(r)$ is a smooth modulus of continuity, which need not satisfy conditions 1), 2) of the paper (^{2}).

Fundamental solutions and solutions of the Cauchy problem for parabolic systems in the sense of Petrovskii under analogous conditions were considered in (^{4}).

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Note: Figure translations are in progress. See original paper for figures.