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## Abstract

## Full Text

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*MATHEMATICS*

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# ON THE CORRECTNESS OF THE FIRST DIFFERENTIAL APPROXIMATIONS OF DIFFERENCE SCHEMES

1. Let $\Lambda(t, x, \tau, h, T)$ be a difference operator approximating the differential operator $\mathcal{L}(t, x, \partial/\partial t, \partial/\partial x)$. Here $x$ is a point of the real $s$-dimensional space $R_s$; $\partial/\partial x = (\partial/\partial x_1, \dots, \partial/\partial x_s)$; $T^1 = (T_1, \dots, T_s)$, $T = (T_0, T^1)$; $T_0$ is the shift operator in $t$; $T_j$ is the shift operator in $x_j$; $\tau, h$ are mesh parameters; $h = (h_1, \dots, h_s)$.

The following operator representation holds:

$$T_0 = e^{\tau \partial/\partial t}, \qquad T_j = e^{h_j \partial/\partial x_j}, \qquad j = 1, \dots, s.$$

Expanding the operator $\Lambda(t, x, \tau, h, T) = \Lambda(t, x, \tau, h, e^{\tau\partial/\partial t}, e^{h\partial/\partial x})$ in a series in the parameters $\tau, h$, we obtain

$$\Lambda(t, x, \tau, h, T) = \mathcal{L}(t, x, \tau, h, \partial/\partial t, \partial/\partial x) + R,$$

$$\mathcal{L}(t, x, \tau, h, \partial/\partial t, \partial/\partial x) = \mathcal{L}(t, x, \partial/\partial t, \partial/\partial x) + \tau^\alpha h^\beta P_{\alpha\beta}(t, x, \partial/\partial t, \partial/\partial x),$$

$$R = \tau^\gamma h^\delta P_{\gamma\delta}(t, x, \partial/\partial t, \partial/\partial x).$$

Summation is carried out over the repeated index; $\tau^\alpha h^\beta P_{\alpha\beta}(t, x, \partial/\partial t, \partial/\partial x)$ are the first terms of the expansion (the lowest in powers of $\tau$ and $h$).

We shall call the operator $\mathcal{L}(t, x, \tau, h, \partial/\partial t, \partial/\partial x)$ the first differential approximation of the difference operator $\Lambda(t, x, \tau, h, T)$, and the equation $\mathcal{L}(t, x, \tau, h, \partial/\partial t, \partial/\partial x)u = 0$ the first differential approximation of the difference equation $\Lambda(t, x, \tau, h, T)u = 0$.

2. Consider the Cauchy problem for a hyperbolic system of first-order equations:

$$\frac{\partial u}{\partial t} = \sum_{k=1}^{s} A_k(x, t) \frac{\partial u}{\partial x_k}, \tag{1}$$

$$u(x, 0) = u_0(x), \tag{2}$$

where $u = u(x,t)$ is a vector-valued function with $m$ components; $x = (x_1, \dots, x_s)$ is a point of the real $s$-dimensional space $R_s$; $A_k(x,t)$ are real $m \times m$ matrices.

We approximate system (1) by the difference scheme

$$u^{n+1}(x) = \sum_{\alpha} B_\alpha u^n(x + \tau\lambda_\alpha). \tag{3}$$

For approximation, the consistency conditions must be satisfied:

$$\sum_{\alpha} B_\alpha = I, \qquad \sum_{\alpha} \lambda_\alpha^k B_\alpha = A_k, \qquad k = 1, \dots, s. \tag{4}$$

Here $I$ is the identity matrix; $B_\alpha$ are real $m \times m$ matrices; $\lambda_\alpha = (\lambda_\alpha^1, \dots, \lambda_\alpha^s)$ are displacement vectors.

The first differential approximation of the difference scheme (3) has the form

$$\frac{\partial u}{\partial t} = \sum_{k,l=1}^{s} C_{kl} \frac{\partial^2 u}{\partial x_k \, \partial x_l} + \sum_{k=1}^{s} D_k \frac{\partial u}{\partial x_k}, \tag{5}$$

$$C_{kl} = \frac{\tau}{2} \left[ \sum_{\alpha} \lambda_\alpha^k \lambda_\alpha^l B_\alpha - A_{kA} l \right],$$

$$D_k = \dot{A}_k - \frac{\tau}{2} \left[ \frac{\partial A_k}{\partial t} + \sum_{l=1}^{s} A_l \frac{\partial A_k}{\partial x_l} \right].$$

Here and below, for hyperbolic systems the first differential approximation is considered on the solution.

**Theorem 1.** *In the case of constant coefficients, the Cauchy problem for the first differential approximation (5) of the difference scheme (3), whose coefficients are symmetric positive definite matrices, is well posed.*

Well-posedness is understood in the sense of I. G. Petrovskii (1).

We note that the scheme is stable by virtue of the criterion of K. O. Friedrichs (2).

The difference scheme (3) is called *simple* (3) if the number of displacement vectors is equal to the number of independent variables.

**Lemma.** *If the matrices $A_k$ are symmetric, then the coefficients of a simple scheme are symmetric matrices.*

3. We approximate the system

$$\partial u/\partial t = A(x,t)\partial u/\partial x \tag{6}$$

by the simple scheme

$$u^{n+1}(x) = \sum_{\alpha=1}^{2} B_\alpha u^n(x+\tau\lambda_\alpha). \tag{7}$$

Here $A(x,t)$ is a real symmetric $m \times m$-matrix. Its first differential approximation is

$$\partial u/\partial t = C\,\partial^2 u/\partial x^2 + D\,\partial u/\partial x, \tag{8}$$

$$C = \frac{\tau}{2}\left[\sum_{\alpha=1}^{2} \lambda_\alpha^2 B_\alpha - A^2\right], \qquad D = A - \frac{\tau}{2}\left[\frac{\partial A}{\partial t} + A\frac{\partial A}{\partial x}\right].$$

**Theorem 2.** *In order that the simple difference scheme (7), in the case of constant coefficients, be stable, it is necessary and sufficient that the Cauchy problem for its first differential approximation be well posed.*

**Theorem 3.** *In the case of variable coefficients, if the matrix $A$ is Lipschitz continuous, then well-posedness of the first differential approximation (8) of the simple scheme (7) is a sufficient condition for stability of the scheme.*

The three-point difference scheme approximating system (6),

$$u^{n+1}(x) = \sum_{\alpha=-1}^{1} B_\alpha u^n(x+\alpha h),$$

$$\sum_{\alpha=-1}^{1} B_\alpha = I, \qquad \sum_{\alpha=-1}^{1} \alpha B_\alpha = rA, \qquad r = \frac{\tau}{h}, \tag{9}$$

will be called *majorant* ($^{4,5}$) if

$$B_1 = rA^+, \qquad B_{-1} = -rA^-,$$

$$A = A^+ + A^-, \qquad A^+ \geqslant 0, \qquad A^- \leqslant 0.$$

The first differential approximation of scheme (9) has the form (8), where

$$C = \frac{h^2}{2\tau}(I - B_0)B_0.$$

**Theorem 4.** For the stability of a majorant scheme in the case of constant coefficients, it is necessary and sufficient that its first differential approximation be well posed.

**Theorem 5.** In the case of variable coefficients, if the matrix $A$ is Lipschitz-continuous, then the well-posedness of the first differential approximation of the majorant scheme is a sufficient condition for the stability of the scheme.

**Theorem 6.** In order that a stable three-point scheme (9) in the case of constant coefficients be a majorant one, it is necessary and sufficient that its coefficients be pairwise permutable symmetric matrices and that $B_1 B_{-1} = 0$. We note that in the case of necessity, stability is not required.

**Corollary.** If the coefficients of the three-point difference scheme (9) in the case of constant coefficients are pairwise permutable symmetric matrices and $B_1 B_{-1} = 0$, then for the stability of the scheme it is necessary and sufficient that its first differential approximation be well posed.

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*Note: Figure translations are in progress. See original paper for figures.*

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