ON THE CONNECTED COMPONENTS OF REAL ELLIPTIC SYSTEMS IN THE PLANE

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Abstract

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ON THE CONNECTED COMPONENTS OF REAL ELLIPTIC SYSTEMS IN THE PLANE

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The differential operator:

$$I(u) = \sum_{i=0}^{n} A_i \frac{\partial^n u(x;y)}{\partial x^{n-i} \partial y^i} + \sum_{p+q<n} B_{pq} \frac{\partial^{p+q} u(x;y)}{\partial x^p \partial y^q}, \tag{1}$$

where A_i, B_{pq} are given real $m \times m$-matrices; $u(x;y)$ is an m-dimensional vector

$$u(x;y) = (u_1(x;y); u_2(x;y), \dots, u_m(x;y)),$$

is called **elliptic** if the corresponding matrix pencil $L(\lambda)$ of the form

$$L(\lambda) = A_0\lambda^n + A_1\lambda^{n-1} + \dots + A_n \tag{2}$$

satisfies the condition

$$\det A_0 \neq 0; \qquad \det L(\lambda) \neq 0, \quad -\infty < \lambda < +\infty. \tag{3}$$

The pencil (2), when condition (3) is fulfilled, will be called **elliptic**.

Two elliptic operators are said to belong to the same connected component if the corresponding matrix pencils belong to the same connected component, i.e., if the pencils

$$L(\lambda) = A_0\lambda^n + A_1\lambda^{n-1} + \dots + A_n,$$
$$L'(\lambda) = A'_0\lambda^n + A'_1\lambda^{n-1} + \dots + A'_n$$

can be deformed into one another in the class of real elliptic pencils by means of a continuous deformation of the matrices $A_0, A_1, \dots, A_n$ into the matrices $A'_0, A'_1, \dots, A'_n$, respectively. In the present paper the question of the number of connected components of the set of real elliptic operators is considered. This problem was posed in a joint report by I. M. Gel' fand, I. G. Petrovskii, and G. E. Shilov at the Third Mathematical Congress in 1956 (see ([1])). B. Boyarsky succeeded in solving it for the case of matrix coefficients of second order, $m = 2$ (see ([2])). According to B. Boyarsky' s theorem, elliptic operators of the form

(1) with $m = 2$ split into $2(n+1)$ connected components. In the present paper a solution of the problem is given for arbitrary $m > 2$. It turns out that in this case all elliptic operators split into four connected components.

In accordance with condition (3), the whole set of real elliptic operators (and the corresponding matrix pencils) splits into two disconnected subsets: $\mathcal{L}_+(n;m)$, when $\det L(\lambda) > 0$, and $\mathcal{L}_-(n;m)$, when $\det L(\lambda) < 0$. In what follows we shall consider only the subset $\mathcal{L}_+(n;m)$, since $\mathcal{L}_-(n;m)$, obviously, has the same number of connected components.

Theorem. *The subset $\mathcal{L}_+(n;m)$ splits into two connected components.*

For n even $(n = 2q,\ m > 2)$, representatives of the components are the operators

$$I_{01} = \Delta^q E_m, \qquad I_{02} = \Delta^{q-1}\begin{pmatrix} \Delta E_{m-2} & 0 \\ 0 & B \end{pmatrix} \tag{4}$$

respectively, where Δ is the Laplace operator: $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, and B is an operator of the form*

$$B = \begin{pmatrix} \partial^2/\partial x^2 - \partial^2/\partial y^2 & 2\partial^2/\partial x\partial y \\ -2\partial^2/\partial x\partial y & \partial^2/\partial x^2 - \partial^2/\partial y^2 \end{pmatrix}.$$

For odd n.** $n = 2q+1$ and $m = 2k,\ k > 1$, the representatives of the connected components are the operators:

$$I_{11} = \Delta^q \begin{pmatrix} K & & 0 \\ & \ddots & \\ 0 & & K \end{pmatrix}, \qquad I_{12} = \Delta^q \begin{pmatrix} K & & 0 \\ & \ddots & \\ 0 & & K' \end{pmatrix}, \tag{5}$$

respectively, where the operators K, K' have the form

$$K = \begin{pmatrix} \partial/\partial x & \partial/\partial y \\ -\partial/\partial y & \partial/\partial x \end{pmatrix}, \qquad K' = \begin{pmatrix} \partial/\partial x & -\partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{pmatrix}.$$

We outline the proof of the theorem. Without loss of generality one may assume that all eigenvalues of the elliptic matrix pencil (2) are simple*** and that any subsystem of its eigenvectors $\psi_1, \psi_2, \dots, \psi_p$ for $p \le m$ is linearly independent. This can be achieved by a small deformation of the pencil which does not take it out of the connected component.

Let the matrices F and D have the form

$$F = (\psi_1 \psi_2 \dots \psi_m), \qquad D = \operatorname{diag}(\lambda_1 \lambda_2 \dots \lambda_m), \tag{6}$$

where ψ_i is an eigenvector of the elliptic pencil $L(\lambda)$ corresponding to the eigenvalue λ_i; then the matrix

$$Z = FDF^{-1} \tag{7}$$

is a right matrix root of the pencil.

Below we shall dwell only on the case of even m, $m = 2k$ with $k > 1$. In this case the pencil $L(\lambda)$ has a real right root z_1, which is constructed with the help of the eigenvectors $\psi_1, \bar{\psi}_1, \dots, \psi_k, \bar{\psi}_k$ and the corresponding eigenvalues $\lambda_1, \bar{\lambda}_1, \dots, \lambda_k, \bar{\lambda}_k$ by formulas (6), (7). Using this fact and Bézout' s theorem (see [4], p. 89), one can decompose the pencil into real linear factors

$$L(\lambda) = A_0(\lambda E + (-1)^n Z_n) \dots (\lambda E + Z_2)(\lambda E - Z_1), \tag{8}$$

where Z_j are real matrices.

The matrices Z_j can be represented in the form

$$Z_j = Q_j G_j Q_j^{-1}, \qquad j = 1, 2, \dots, n, \tag{9}$$

where Q_j, G_j are real matrices, and the G_j have a quasi-diagonal form, with second-order blocks of the form**** $\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$ on the diagonal, where $\beta > 0$ (see [4], p. 257). In the pencil $L(\lambda)$ we deform the matrix A_0 into E_{2k}, preserving nonsingularity, and the matrices Q_j either into E_{2k}, if $\det Q_j > 0$, or into the matrix $E'_{2k} = \mathrm{diag}(1, \dots, 1, -1)$, if $\det Q_j < 0$. The blocks on the diagonal of G_j are deformed into the block having the form $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. At the same time, obviously, the reality and ellipticity of the pencil are preserved. As a result of these transformations the pencil takes the form

$$\tilde{L}(\lambda) = (\lambda E + (-1)^n \tilde{G}_n) \dots (\lambda E + \tilde{G}_2)(\lambda E - \tilde{G}_1), \tag{10}$$

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* Bidzade was the first to indicate this elliptic operator [3].

** In view of ellipticity, with n odd m is necessarily even.

*** Thus, we assume that the pencil has $n \cdot m$ distinct eigenvalues.

**** The values α and β are respectively equal to the real and imaginary parts of an eigenvalue of the matrix Z_j.

where $\widetilde{G}_j$ coincides with one of the matrices of the form:

$$G_0 = \begin{pmatrix} 0 & -1 & & 0 \\ 1 & 0 & & \\ & & \ddots & \\ 0 & & 0 & -1 \\ & & 1 & 0 \end{pmatrix}, \qquad G_0' = \begin{pmatrix} 0 & -1 & & 0 & & & \\ 1 & 0 & & & & & \\ & & \ddots & & & & \\ 0 & & & 0 & -1 & & \\ & & & 1 & & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix}. \tag{11}$$

It is easy to verify the validity of the relation

$$(\lambda E \pm G_0)(\lambda E \mp G_0') = T(\lambda E \pm G_0')(\lambda E \mp G_0)T^{-1}, \tag{12}$$

where the matrix has the form

$$T = \begin{pmatrix} 1 & & 0 & \\ & \ddots & & \\ & & 1 & \\ & & -1 & \\ 0 & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}, \qquad \det T > 0 \tag{13}$$

and, in view of the last inequality, is topologically equivalent to E_{2k}. On the basis of this fact the pencil $\widetilde{L}(\lambda)$ can be deformed to the form:

$$\widetilde{L}(\lambda) = (\lambda E + (-1)^n G_0') \ldots (\lambda E + (-1)^{p+1} G_0')(\lambda E - (-1)^p G_0) \ldots (\lambda E - G_0). \tag{14}$$

Lemma 1. *The subset of elliptic pencils corresponding to the operators $\mathfrak{L}_+(n;m)$, for n, m even, $n = 2q$, $m = 2k$, $k > 1$, splits into two connected components. Representatives of each component are, respectively, the pencils:*

$$L_{01}(\lambda) = (\lambda^2 + 1)^q E_{2k}, \tag{15}$$

$$L_{02}(\lambda) = (\lambda^2 + 1)^{q-1} \begin{pmatrix} \lambda^2 + 1 & & 0 & & \\ & \ddots & & & \\ & & \lambda^2 + 1 & & \\ & & & \lambda^2 - 1 & 2\lambda \\ 0 & & & -2\lambda & \lambda^2 - 1 \end{pmatrix}. \tag{16}$$

respectively.

Indeed, by virtue of the easily verified equalities

$$(\lambda E + G_0')(\lambda E - G_0') = (\lambda^2+1)E_{2k}, \qquad (\lambda E + G_0)(\lambda E - G_0) = \\ = (\lambda^2+1)E_{2k}, \tag{17}$$

in the case of even p the pencil (14) takes the form (15). It is also not difficult to verify directly that, for odd p, the pencil (14) coincides with the pencil (16).

We shall now show that the pencils (15) and (16) are topologically inequivalent (belong to different connected components). Introduce the auxiliary matrix cycles $W_{01}(\varphi)$ and $W_{02}(\varphi)$, $0 \le \varphi \le \pi$ (closed curves in the group of nonsingular matrices), by the formulas:

$$W_{01}(\varphi) = \sin^{2q}\varphi\, L_{01}(\operatorname{ctg}\varphi) = E_{2k}, \qquad 0 \le \varphi \le \pi, \\ W_{01}(0) = W_{01}(\pi) = E_{2k}; \tag{18}$$

$$W_{02}(\varphi) = \sin^{2q}\varphi\, L_{02}(\operatorname{ctg}\varphi) = \begin{pmatrix} 1 & & 0 & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \cos 2\varphi & \sin 2\varphi \\ 0 & & & -\sin 2\varphi & \cos 2\varphi \end{pmatrix}, \qquad 0 \le \varphi \le \pi,$$

$$W_{02}(0) = W_{02}(\pi) = E_{2k}. \tag{19}$$

It is clear that the topological equivalence of the pencils $L_{01}(\lambda)$ and $L_{02}(\lambda)$ in the sense indicated by us at the beginning of the article would imply a homotopy of the cyc-

of $W_{01}(\varphi)$ and $W_{02}(\varphi)$. The latter, however, does not occur, since the cycle $W_{02}(\varphi)$ cannot be contracted to a point (see (5)). This lemma establishes the first half of the theorem in the case $n = 2q$ (m even). In the case of odd n the following assertion is true.

Lemma 2. *The subset of elliptic pencils corresponding to the operators $\mathcal{E}_+(n;m)$ for odd $n = 2q+1$ and even $m = 2k,\ k > 1$, splits into two connected components. Representatives of each component are the pencils:*

$$L_{11}(\lambda) = (\lambda^2+1)^q(\lambda E - G_0), \tag{20}$$

$$L_{12}(\lambda) = (\lambda^2+1)^q(\lambda E - G_0') \tag{21}$$

respectively.

Introduce the matrix curves $W_{11}(\varphi)$ and $W_{12}(\varphi)$, $0 \le \varphi \le \pi$, corresponding to the pencils $L_{11}(\lambda)$ and $L_{12}(\lambda)$ by the formulas

$$W_{11}(\varphi) = \sin^{2q+1}\varphi\, L_{11}(\operatorname{ctg}\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi & & 0 & \\ -\sin\varphi & \cos\varphi & & & \\ & & \ddots & & \\ 0 & & & \cos\varphi & \sin\varphi \\ & & & -\sin\varphi & \cos\varphi \end{pmatrix}, \qquad 0 \le \varphi \le \pi, \tag{22}$$

$$W_{11}(0) = E_{2k}, \qquad W_{11}(\pi) = -E_{2k};$$

$$W_{12}(\varphi) = \sin^{2q+1}\varphi\, L_{12}(\operatorname{ctg}\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi & & 0 & & \\ -\sin\varphi & \cos\varphi & & & & \\ & & \ddots & & & \\ & & & \cos\varphi & \sin\varphi & \\ 0 & & & -\sin\varphi & \cos\varphi & \\ & & & & \cos\varphi & -\sin\varphi \\ & & & & \sin\varphi & \cos\varphi \end{pmatrix},$$

$$0 \le \varphi \le \pi, \qquad W_{12}(0) = E_{2k}, \qquad W_{12}(\pi) = -E_{2k}. \tag{23}$$

These curves have common endpoints; however, they are not homotopic, since the cycle $W_0(\varphi)$

$$W_0(\varphi) = \begin{cases} W_{11}(\varphi), & 0 \le \varphi \le \pi, \\ W_{12}(2\pi - \varphi), & \pi \le \varphi \le 2\pi, \end{cases} \tag{24}$$

is homotopic to the cycle

$$\widetilde{W}_0 = \begin{pmatrix} 1 & & 0 & \\ & \ddots & & \\ 0 & & 1 & \\ & & \cos\varphi & \sin\varphi \\ & & -\sin\varphi & \cos\varphi \end{pmatrix}, \qquad 0 \le \varphi \le 2\pi, \tag{25}$$

which is not contracted to a point. This lemma establishes the second part of the theorem for even m.

If m is odd, then the pencil $L(\lambda)$ has no real matrix root, and its reduction to the canonical form (4) will require more complicated transformations, which we shall consider in a detailed article.

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Note: Figure translations are in progress. See original paper for figures.