# ON MINIMAL HAUSDORFF SPACES AND \(T_1\)-BICOMPACTS

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## Abstract

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### MATHEMATICS

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# ON MINIMAL HAUSDORFF SPACES AND $T_1$-BICOMPACTS

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In the present note we shall prove the following theorems.

**Theorem 1.** Let the Hausdorff space $(X, \mathcal{T})$ be the sum of a countable set of its bicompact nowhere dense subsets $\Phi_n$, $n = 1, 2, \ldots$. Suppose, in addition, that the space $(X, \mathcal{T})$ is a dense subspace of a space $(Y, \mathcal{T}')$ which has the Baire property (in it there is no countable covering by closed nowhere dense subsets). Then the space $(X, \mathcal{T})$ cannot be strengthened to a minimal Hausdorff space.

**Theorem 2.** Let the spaces $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ satisfy the assumptions of Theorem 1 and let, in addition, the space $(X, \mathcal{T})$ be a $k$-space. Denote by $\mathcal{T}_k$ the set of all subsets of the form $X \setminus A$, where $A$ is any bicompact subset in the space $(X, \mathcal{T})$.

Then:

(a) $(X, \mathcal{T}_k)$ is a $T_1$-bicompact.

(b) If $\mathcal{T}_k \subseteq \mathcal{T}_1$ and the space $(X, \mathcal{T}_1)$ is a $T_1$-bicompact, then $\mathcal{T}_1 \subseteq \mathcal{T}$.

Thus, by Theorem 1, we obtain

(c) There exists no Hausdorff bicompact topology $\mathcal{T}_1$ which majorizes the topology $\mathcal{T}_k$.

From Theorem 1 there immediately follows

**Theorem** $1'$**.** If a completely regular space is the sum of a countable set of its nowhere dense bicompact subsets, then it cannot be strengthened to a minimal Hausdorff space.

In the same way, adding to the space $(X, \mathcal{T})$ the assumption that it is completely regular, and saying nothing about the space $(Y, \mathcal{T}')$, we obtain from Theorem 2 a certain theorem $2'$. In particular, we obtain the following example.

**Example.** Let $(X, \mathcal{T})$ be the space of rational numbers with the ordinary topology. Then, by Theorem 1, this space cannot be strengthened to a minimal

Hausdorff space (see (4)). Moreover, by Theorem 2, the space $(X, \mathcal{T}_k)$ is a $T_1$-bicompact whose topology is not majorized by any minimal Hausdorff topology. It is interesting that the space $(X, \mathcal{T}_k)$ has a countable net (see (1)), consisting of closed (simply one-point) subsets, but there is no countable base in this space.

Thus, in contrast to Hausdorff bicompacts, $T_1$-bicompacts in general are not $A$-spaces (see (4)), and even their weight may exceed their cardinality.

Another example of the same kind* is given in (2) (see § 9). In the same paper (2) A. Arhangel' skii asks: whether every

* In (2) there is an example of a countable, non-Hausdorff $T_1$-bicompact which is not a strengthening of any $T_1$-bicompact distinct from itself.

is a $T_1$-bicompact space with a countable base a compactification of some Hausdorff bicompactum? A negative answer to this question is given by the following

**Theorem 3.** *There exists a $T_1$-bicompactum with a countable base which is not a compactification of any Hausdorff bicompactum.*

**Remark.** In [2] it is said that the following theorem is well known—it is almost trivial, but quite effective.

A space with the first axiom of countability, every bicompact Hausdorff subspace of which is closed, is a Hausdorff space.

Hence we obtain that any non-Hausdorff $T_1$-bicompactum with a countable base is a nontrivial compactification of some other bicompactum.

**Proof of Theorem 1.** Let $U_y$ be the filter in the space $X$ of intersections of neighborhoods of the point $y \in Y$ with the set $X$. This is a filter with an open base. Let us now denote, for some fixed but arbitrary Hausdorff topology $\mathcal{T}_0 \subseteq \mathcal{T}$, defined on the set $X$, by $A_n$ the set of all points $y$ of the space $Y$ for which there exists in $\Phi_n$ a point of contact of the filter $U_y$, $n = 1, 2, \ldots$. The sets $A_n$ are nowhere dense in the space $Y$, since

$$A_n \cap (X \setminus \Phi_n) = \emptyset, \qquad n = 1, 2, \ldots .$$

Moreover, they are closed in the space $Y$.

Indeed, if $y \notin A$, then in the space $(X, \mathcal{T}_0)$ there exists some open cover of the bicompact set $\Phi_n$, and therefore also a finite open cover of this set, by sets which do not meet certain sets of the filter $U_y$. Thus there exists one common element $O_y \cap X$ of the filter $U_y$ which does not meet any set of this finite open cover of the set $\Phi_n$. This means that $A_n \cap O_y = \emptyset$; hence $A_n$ is a closed set in the space $Y$.

Since the space $Y$ has the Baire property, the sets $A_1, A_2, \ldots$ do not form a cover, i.e. for some filter $U_y$, $y \in Y$, there is no point of contact in $X$. Thus the Hausdorff space $(X, \mathcal{T}_0)$ is not even absolutely closed, in particular not minimal* (see [3], I, §10, Exercises 10 and 11). The theorem is proved.

**Proof of Theorem 2.** (a) is obvious.

(b). Let $\Phi$ be a closed subset of the space $(X, \mathcal{T}_1)$. Then the intersection of the set $\Phi$ with any bicompact subset $B$ of the space $(X, \mathcal{T})$ is closed in this space.

Indeed, by definition of the topology $\mathcal{T}_k$, the set $B$ is a closed subset of the space $(X, \mathcal{T}_k)$, and therefore also of the space $(X, \mathcal{T}_1)$. Thus $B$ is a bicompact subset of the space $(X, \mathcal{T}_1)$. Obviously,

$$\mathcal{T}|B = \mathcal{T}_k|B$$

(this is how the topologies induced on subsets are denoted). Hence $(B, \mathcal{T}_k|B)$ is a Hausdorff bicompactum which is a compactification of the bicompactum $(B, \mathcal{T}_1|B)$. Therefore this other bicompactum is also Hausdorff, and they are homeomorphic to one another. Thus we obtain that $\Phi \cap B$ is closed in the space $(X, \mathcal{T})$ for every bicompact subset of this space. But the space $(X, \mathcal{T})$ is a $k$-space; consequently, $\Phi$ is a closed subset of the space $(X, \mathcal{T})$. The theorem is proved.

**Proof of Theorem 3.** Let $X$ be the set of all rational numbers. We define a topology $\mathcal{T}$ as the set of all subsets $\Gamma \subseteq X$ such that:

(1) the set $X \setminus \Gamma$ is contained in a finite sum of sets of the form

$$\{x\} \cup \{x+1,\ x + {}^1/_2,\ x + {}^1/_3, \dots\}, \qquad x \in X,$$

(2) together with any of its elements $x$, the set $\Gamma$ contains almost all numbers of the form $x+1,\ x + {}^1/_2,\ x + {}^1/_3, \dots$

From (1) and (2) it follows that sets of the form

$$\{y\} \cup \left[ X \setminus \bigcup_{i=1}^{n} (\{x_i\} \cup \{x_i + 1,\ x_i + {}^1/_2,\ x_i + {}^1/_3, \dots\}) \right],$$

where $Y = \{x_1, \dots, x_n\}$, constitute a base of the topology $\mathcal{T}$. Obviously, this base is countable.

The space $(X, \mathcal{T})$ is a $T_1$-bicompactum. Indeed, any closed subset of the space $(X, \mathcal{T})$ distinct from the whole space is a Hausdorff bicompactum with the topology induced by the usual topology of the space of rational (or real) numbers. Therefore the intersection of any centered system of closed subsets of the space $(X, \mathcal{T})$ is nonempty.

Now let the topology $\mathcal{T}_1$ majorize the topology $\mathcal{T}$. If the space $(X, \mathcal{T}_1)$ is a $T_1$-bicompactum, then every point in $X$ remains a condensation point, since

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* An absolutely closed space is compactified to a minimal one in a unique way, so that in fact no substantial generalization is obtained.

otherwise we would obtain an infinite, discrete and closed set of the form $\{x+1,\ x+{}^1/_2,\ x+{}^1/_3,\ldots\}$, contrary to the bicompactness of the space $(X,\mathcal{T}_1)$.

Thus every one-point subset of the space $(X,\mathcal{T}_1)$ is closed and nowhere dense. Consequently, if the space $(X,\mathcal{T}_1)$ is bicompact, then it is not Hausdorff (Hausdorff bicompacta have the Baire property). The theorem is proved.

**Remark.** All the non-Hausdorff $T_1$-bicompacta indicated in this note do not possess the Baire property. But the simplest space of this kind is, of course, the countably infinite space with the smallest $T_1$-topology.

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*Note: Figure translations are in progress. See original paper for figures.*