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Abstract

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MATHEMATICS

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ON AXISYMMETRIC FLOWS WITH A FREE BOUNDARY

(Presented by Academician I. N. Vekua on 29 V 1967)

1. Formulation of the problem. An axisymmetric vector field is determined by two functions $V_x(x, y)$, $V_y(x, y)$, defined in some domain in the meridian plane (x, y) , $y \geq 0$. The absence, for the field (V_x, V_y) , of sources (sinks) and vorticity at some point (x, y) is written in the form of two equations

$$\frac{\partial}{\partial x}(yV_x) + \frac{\partial}{\partial y}(yV_y) = 0, \quad \frac{\partial}{\partial x}V_y - \frac{\partial}{\partial y}V_x = 0. \quad (1)$$

Introducing the notation $w = V_y + iV_x$, $\partial/\partial z = (\partial/\partial x + i\partial/\partial y)/2$, we write system (1) in the form of a single complex equation

$$\partial w/\partial \bar{z} - w/4iy - \bar{w}/4iy = 0. \quad (2)$$

In this paper the following problem is investigated: it is required to find a closed simple (sufficiently smooth) curve γ (the "free boundary"), located in the upper half-plane $y > 0$, such that in the domain G bounded by the curve γ there exists a solution w of equation (2) satisfying the following three conditions:

- a) at some (fixed) interior point z_0 of the domain G , the solution w has an isolated pole:

$$|w(z_0)| = \infty; \quad (3)$$

- b) on the set $(G + \gamma) \setminus z_0$, the function w is regular and satisfies the flow-tangency condition

$$\operatorname{Re}[w dz] \equiv V_y dx - V_x dy = 0, \quad z \in \gamma; \quad (4)$$

- c) on the free boundary γ , a constant pressure is maintained,

$$|w(z)| = R, \quad R = \text{const} \neq 0, \quad z \in \gamma \quad (5)$$

(Bernoulli's law for a weightless ideal incompressible fluid).

2. Reduction to an auxiliary equivalent problem. Suppose for the time being that the problem under consideration has a (nonzero) solution $w(z)$. Since in the domain G the coefficients of equation (2) are regular, for any solution of the problem (2), (4) the following relation of I. N. Vekua holds: $2N_G + N_\gamma - 2P_G - P_\gamma = 2\chi$, where N_G is the total multiplicity of the interior zeros of the field $w(z)$; N_γ is the total multiplicity of zeros of the field on the boundary γ ; P_G, P_γ are the analogous multiplicities of the poles; χ is the index of the boundary condition (see ⁽¹⁾, Chap. IV, § 4). In the case under consideration $\chi = -1, P_\gamma = 0, N_\gamma = 0$, so that $N_G - P_G = -1$. Assuming that the field w has no zeros inside G , we arrive at the conclusion that $P_G = 1$, i.e., the solution $w(z)$ has at the point z_0 a pole of the first order. Hence follows the one-to-one character of the mapping $w = w(z)$ (at least in a neighborhood of the point $z = z_0$). Taking also condition (5) into account, it is now easy to verify that the function $\tau = 1/w(z)$ effects a one-to-one mapping of the domain G onto the circle of radius $r = 1/R$ with center at the point $\tau = 0$. We shall therefore seek

the function $z = z(\tau) \equiv z(1/w)$, inverse to the function $\tau = \tau(z) = 1/w(z)$. It is not difficult to verify that the Jacobian of the mapping $w = w(z)$ is positive. As direct computations show, the function $z = z(\tau)$, by virtue of equation (2), satisfies the nonlinear equation

$$\frac{\partial z}{\partial \tau} = \text{sign Re } \tau \cdot \frac{-i|\partial z/\partial \tau|^2}{y/|\text{Re } \tau| + \sqrt{y^2/|\text{Re } \tau|^2 + |\partial z/\partial \tau|^2}}. \quad (6)$$

The solution of this equation must be defined in the disk $\bar{K} : |\tau| \leq r, r = 1/R$, and must satisfy, by virtue of relation (4), the boundary condition

$$\text{Re} [e^{-i\varphi} dz/d\varphi] = 0 \quad \text{on the circle } |\tau| = r \quad (7)$$

($\tau = |\tau|e^{i\varphi}$). Finally, without loss of generality the point z_0 may be placed on the imaginary axis, so that $z_0 = ih, h > 0$. Condition (3) for the function $z(\tau)$ corresponds to

$$z(0) = ih. \quad (8)$$

Below we shall investigate the problem (6), (7), and (8).

3. Spaces and operators. Let C_α be the set of bounded complex-valued functions $\omega(\tau)$, defined in the disk \bar{K} and satisfying the Hölder condition with exponent $\alpha, 0 < \alpha < 1$. As usual, for functions $\omega(\tau)$ from C_α we introduce the norm

$$\|\omega\|_\alpha = \|\omega\|_0 + \sup_{\tau_1 \neq \tau_2} \frac{|\omega(\tau_1) - \omega(\tau_2)|}{|\tau_1 - \tau_2|}, \quad \|\omega\|_0 = \max_{\tau \in \bar{K}} |\omega(\tau)|,$$

which turns the set C_α into a Banach space. It is easy to see that for any two functions from C_α the inequality

$$\|\omega\sigma\|_\alpha \leq \|\omega\|_0 \|\sigma\|_\alpha + \|\sigma\|_0 \|\omega\|_\alpha. \quad (9)$$

holds.

To study the problem (6), (7), (8), we shall apply the method of two-dimensional singular integral operators developed in the investigations of I. N. Vekua (see ⁽¹⁾, Ch. IV, § 9). To this end we first study the problem of the general representation of all continuous and continuously differentiable complex-valued functions $z = z(\tau)$ satisfying conditions (7) and (8). As the investigation shows, such a general representation has the form

$$z(\tau) = -\frac{1}{\pi} \iint_K \left[\frac{r^2 \tau \rho(\zeta)}{\zeta(\zeta - \tau)} + \frac{\tau^3 \overline{\rho(\zeta)}}{r^2 - \tau \bar{\zeta}} \right] dK_\zeta + \frac{2}{\pi} \int_0^\tau \left\{ \iint_K \frac{\tau_1^2 \overline{\rho(\zeta)}}{r^2 - \tau_1 \bar{\zeta}} dK_\zeta \right\} d\tau_1 + c_0 r^2 \tau + ih \equiv T\rho, \quad (10)$$

where ρ is a completely arbitrary function from the space $C_\alpha(\bar{K})$, and c_0 is an arbitrary real constant. In this case, as is known, the formulas

$$\frac{\partial z}{\partial \bar{\tau}} = r^2 \rho(\tau), \quad \frac{\partial z}{\partial \tau} = r^2 \left[-\frac{1}{\pi} \iint_K \frac{\rho(\zeta)}{(\zeta - \tau)^2} dK_\zeta - \frac{1}{\pi} \iint_K \frac{\bar{\tau}^2 \rho(\zeta)}{(r^2 - \bar{\tau} \zeta)^2} dK_\zeta + c_0 \right] \equiv \Pi\rho; \quad (11)$$

hold; the first integral in the second formula is understood in the sense of the Cauchy principal value. If, therefore, the solution of the problem (6), (7), (8) is sought in the form (10) with unknown density $\rho(\tau) \in C_\alpha(\bar{K})$, then, to determine the function $\rho(\tau)$, as follows from formulas (10), (11) and equation (6), we obtain the nonlinear singular integral equation

$$\rho(\tau) = \frac{-ie^{2i\varphi} \operatorname{Re} \tau \bar{\Pi} \rho \Pi \rho}{r^2 \left[|\operatorname{Im} T\rho| + \sqrt{|\operatorname{Im} T\rho|^2 + |\operatorname{Re} \tau \Pi \rho|^2} \right]} \equiv S\rho. \quad (12)$$

This equation, for an arbitrary constant c_0 (see (10)), is equivalent to the problem (6), (7), (8).

4. **Property of the singular operator $S\rho$.** In the present section it will be shown that the operator $S\rho$, defined by formula (12), transforms elements of the space $C_\alpha(K)$ into elements of the same space and satisfies a Lipschitz

condition, provided that the radius r of the circle K is sufficiently small, i.e., provided that the number R in condition (5) is sufficiently large.

Consider in the space C_α the ball of radius Q : $\|\rho\|_\alpha \leq Q$. Since all terms of the operator $T\rho$, except the last one ih , tend to zero as $r \rightarrow 0$, uniformly in the fixed ball $\|\rho\| \leq Q$, for arbitrary m , $0 < m < h$, there exists an $r_0 = r_0(h, m, Q)$ such that

$$|y| = |\operatorname{Im} z(\tau)| = |\operatorname{Im} T\rho| > m, \quad (13)$$

if $r \leq r_0$. Further, the operator $\Pi\rho$ maps $C_\alpha(K)$ into $C_\alpha(\bar{K})$ and, as direct computations show,

$$\|\partial z/\partial \tau\|_\alpha = \|\Pi\rho\|_\alpha \leq r^2[M_1\|\rho\|_\alpha + |c_0|] + rM_2\|\rho\|_\alpha, \quad (14)$$

where M_1, M_2 (as well as the constants M_3, M_4 occurring below) depend only on h, m, Q, α and, possibly, on the constant c_0 . From the displayed formulas and certain obvious additional computations there follow the above-noted properties of the operator (12), moreover

$$\|S\rho_1 - S\rho_2\|_\alpha \leq r^2M_3\|\rho_1 - \rho_2\|_\alpha. \quad (15)$$

In addition, if θ is the zero element of the space C_α , we have

$$S\theta = \frac{ie^{2i\varphi}r^2c_0^2 \operatorname{Re} \tau}{|r^2c_0 \operatorname{Im} \tau + h| + \sqrt{|r^2c_0 \operatorname{Im} \tau + h|^2 + r^2c_0^4(\operatorname{Re} \tau)^2}},$$

whence we obtain

$$\|S\theta\|_\alpha \leq \frac{r^{3-\alpha}c_0^2}{h - r^3|c_0|} (r^\alpha + 2M_4). \quad (16)$$

It follows from estimates (15), (16) that the numbers c_0 and $r \leq r_0$ can be chosen so that the inequalities

$$r^2M_3 \leq \beta < 1, \quad \|S\theta\|_\alpha < (1 - \beta)Q, \quad \beta = \text{const.} \quad (17)$$

are satisfied. In this case the operator $S\rho$ maps the ball $\|\rho\|_\alpha \leq Q$ into itself and, by the generalized contraction mapping principle, has a fixed point. For a fixed value of the constant c_0 , this fixed point is the unique solution of equation (12), which, as is easy to see, tends to zero simultaneously with the constant c_0 .

5. **Existence theorem.** *Problem (6), (7), (8) always has a solution if the radius r of the circle K is sufficiently small. This solution is representable by formula (10), in which $\rho(\tau)$ is a solution of equation (12), and depends on one real parameter c_0 . Consequently, the original problem (2), (3), (4), (5), with fixed $h > 0$ and sufficiently large R , has a family of solutions depending on one real parameter. The free boundary γ is at least a Lyapunov curve.*

In the proof it remains only to justify the univalence of the mapping effected by the function $z(\tau)$ constructed by formula (10). In view of (6), this function satisfies an equation of Beltrami type:

$$\frac{\partial z}{\partial \bar{\tau}} - q(\tau) \frac{\partial z}{\partial \tau} = 0, \quad \text{where} \quad q(\tau) = \frac{i \operatorname{Re} \tau \partial \bar{z} / \partial \bar{\tau}}{y + \sqrt{y^2 + |\operatorname{Re} \tau \partial z / \partial \tau|^2}}.$$

Since, obviously, $|q(\tau)| \leq q_0 < 1$, $q_0 = \text{const}$, the function $z(\tau)$, as follows from the formula for the Jacobian of the mapping $J = |z_\tau|^2 - |z_{\bar{\tau}}|^2$, is locally univalent in a neighborhood of every point where $\partial z / \partial \tau \neq 0$. From formula (11) and by decreasing, if necessary, the radius r , we obtain $\partial z / \partial \tau \neq 0$ at the point $\tau = 0$.

Finally, let us note that, as can be shown from condition (7), γ is a convex curve.

In conclusion I express my sincere gratitude to I. I. Danilyuk for posing the problem and for valuable advice.

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CITED LITERATURE

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2. N. E. Kochin, I. A. Kibel, N. V. Roze, *Theoretical Hydromechanics*, Part I, Moscow, 1963.

Note: Figure translations are in progress. See original paper for figures.

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