ON MONOTONE SOLUTIONS OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS OF THE n-TH ORDER

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Abstract

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ON MONOTONE SOLUTIONS OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS OF THE n-TH ORDER

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In the present paper we study the problem of the existence of a solution $u(t)$ of the differential equation

$$u^{(n)} = f(t, u, u', \dots, u^{(n-1)}), \tag{1}$$

defined on the interval $(0, +\infty)$ and satisfying the conditions

$$\lim_{t\to 0} u(t) = u_0, \qquad (-1)^k u^{(k)}(t) \geq 0 \quad \text{for } t > 0 \ (k = 0, 1, \dots, n-1). \tag{2}$$

Previously this problem had been investigated only for the case $n = 2$ ([1–5]).

By a solution of problem (1)–(2) we mean a function $u(t)$, absolutely continuous together with its derivatives up to order $n-1$ inclusive on every segment contained in the interval $(0, +\infty)$, and satisfying equation (1) and conditions (2) on this interval.

Before proceeding to the formulation of the existence theorem, we give some definitions and one auxiliary proposition.

Definition 1. $\omega(t, x) \in B_n(r; a)$ if $\omega(t, x)$ is defined and nonnegative in the domain $t \in (0, a]$, $x \in [0, +\infty)$, and there exist a number $a_0 \in (0, a)$ and a function $b(t)$, continuous on the half-interval $(0, a]$, with $t^{n-2} b(t) \in L(0, a)$, such that, for any $\alpha \in (0, a_0]$, for every function $u(t)$ absolutely continuous together with its derivatives up to order $n-1$ inclusive on the interval $[\alpha, a]$ and satisfying on this interval the inequalities

$$(-1)^k u^{(k)}(t) \geq 0 \quad (k = 0, 1, \dots, n), \qquad u(t) \leq r, \qquad |u^{(n)}(t)| \leq \omega(t, |u^{(n-1)}(t)|),$$

we have $|u^{(n-1)}(t)| \leq b(t)$ for $t \in [\alpha, a]$.

Definition 2. $\psi(t, x_1, \dots, x_m) \in K_t(\alpha, \beta)$ if $m \geq 1$, $\psi(t, x_1, \dots, x_m)$ is measurable in t on the interval (α, β) for every $(x_1, \dots, x_m) \in R_m$, continuous in

$(x_1, \dots, x_m)$ in the space R_m for almost all $t \in (\alpha, \beta)$, and for every $r > 0$ there exists a function $\psi(t; r) \in L(\alpha, \beta)$ such that

$$|\psi(t, x_1, \dots, x_m)| \le \psi(t; r)$$

for $t \in (\alpha, \beta)$, $|x_k| \le r$ $(k = 1, 2, \dots, m)$ (local Carathéodory conditions).

In what follows it is everywhere assumed that $n \ge 2$ and $f(t, x_1, \dots, x_n) \in K_t(\alpha, \beta)$ for arbitrary $\alpha, \beta \in (0, +\infty)$; here the case is not excluded when the function $f(t, x_1, \dots, x_n)$, having a singularity at $t = 0$, does not belong to the set $K_t(0, \beta)$.

Lemma. If $n \ge 3$, the function $u(t)$ is absolutely continuous together with its derivatives up to order $n - 1$ inclusive on the interval $[\alpha, \beta]$, $u^{(k)}(\beta) = 0$ $(k = 1, \dots, n - 2)$, and

$$(-1)^k u^{(k)}(t) \ge 0$$

for $t \in [\alpha, \beta]$ $(k = 0, 1, \dots, n)$, then for $t \in [\alpha, \beta]$ the inequalities

$$|u^{(k-1)}(t)| \le \frac{[(n-1)!]^{(n-k)/(n-1)}}{(n-k)!} |u(t)|^{(n-k)/(n-1)} |u^{(n-1)}(t)|^{(k-1)/(n-1)}$$

$$(k = 1, 2, \dots, n).$$

Theorem 1. If

$$f(t, 0, \dots, 0) \equiv 0, \qquad (-1)^n f(t, x_1, \dots, x_n) \ge 0$$

for $t > 0$, $(-1)^{k-1} x_k \ge 0$ $(k = 1, 2, \dots, n)$,
$a > 0$, $r > 0$, and in the domain

$$t \in (0, a], \qquad 0 \le x_1 \le r, \qquad 0 \le (-1)^{k-1} x_k \le$$

$$\le \frac{[(n-1)!\, x_1]^{(n-k)/(n-1)}}{(n-k)!} |x_n|^{(k-1)/(n-1)}$$

$$(k = 2, \dots, n-1), \qquad (-1)^{n-1} x_n \ge 0 \tag{4}$$

the inequality

$$|f(t, x_1, \dots, x_n)| \le \omega(t, |x_n|) \tag{5}$$

is satisfied, where $\omega(t, x) \in B_n(r; a)$, then for every $u_0 \in [0, r]$ the problem (1)–(2) has at least one solution.

We indicate the scheme of the proof of Theorem 1. Let the number a_0 and the function $b(t)$ be chosen according to Definition 1. Put

$$b_k(t) = (n-1)!\, r\, (a-a_0)^{-k} + \int_t^a \tau^{\,n-2-k} b(\tau)\, d\tau \quad (k = 0, \dots, n-2),$$

$$b_{n-1}(t) = b(t); \tag{6}$$

$$\rho(t) = \max_{t \le \tau \le a} \sum_{k=0}^{n-1} b_k(\tau); \tag{7}$$

$$\chi(t;\tau) = \begin{cases} 1, & \text{for } 0 \le t \le \tau, \\ 2 - t/\tau, & \text{for } \tau < t < 2\tau, \\ 0, & \text{for } t \ge 2\tau; \end{cases} \qquad \sigma_k(t) = \begin{cases} t, & \text{for } (-1)^{k-1} t > 0, \\ 0, & \text{for } (-1)^{k-1} t \le 0; \end{cases} \tag{8}$$

$$g_m(t, x_1, \dots, x_n) = \chi\left(\sum_{i=1}^n |x_i|; \rho\left(\frac{a_0}{m}\right) \right) f(t, \sigma_1(x_1), \dots, \sigma_n(x_n)). \tag{9}$$

Since $t^{n-2} b(t) \in L(0,a)$, it is clear from (6) that

$$b_1(t) \in L(0,a). \tag{10}$$

Taking into account conditions (3), with the aid of Schauder' s theorem it is easy to prove that, for every natural number m, the differential equation

$$u^{(n)} = g_m(t, u, u', \dots, u^{(n-1)}) \tag{11}$$

has a solution $u_m(t)$ satisfying the boundary conditions

$$u_m(a_0/m) = u_0, \qquad u_m^{(k)}(a+m) = 0 \quad (k = 0, 1, \dots, n-2) \tag{12}$$

and the inequalities

$$(-1)^k u_m^{(k)}(t) \ge 0 \quad \text{for } t \in \left[\frac{a_0}{m}, a+m\right] \quad (k = 0, 1, \dots, n). \tag{13}$$

Since $u_0 \in [0, r]$, according to (12) and (13), from the lemma cited above it is clear that, for $t \in [a_0/m, a+m]$, the point $(t, u(t), \dots, u^{(n-1)}(t))$ belongs to the domain (4); therefore, by virtue of (5), (8), and (9), from (11) we have

$$|u_m^{(n)}(t)| \le \omega(t, |u_m^{(n-1)}(t)|) \quad \text{for } t \in [a_0/m, a].$$

Hence, according to Definition 1 and conditions (6), (7), (10), (12), and (13), we easily find that

$$|u_m^{(k)}(t)| \le b_k(t) \quad \text{for } t \in \left[\frac{a_0}{m}, a\right] \quad (k = 0, 1, \dots, n-1),$$

$$\sum_{k=0}^{n-1} |u_m^{(k)}(t)| \le \rho(a_0/m) \quad \text{for } t \in \left[\frac{a_0}{m}, a+m\right]; \tag{14}$$

$$|u_m(t) - u_0| \le \int_0^t b_1(\tau)\, d\tau \quad \text{for } t \in \left[\frac{a_0}{m}, a+m\right]. \tag{15}$$

By virtue of (8), (9), and (14), it follows from (11) that $u_m(t)$ is a solution of equation (1) on the interval $[a_0/m, a+m]$.

Relying on inequalities (13) and (14), one can show that the sequence $\{u_m(t)\}$ contains a subsequence $\{u_{m_i}(t)\}$ such that $\{u_{m_i}^{(k)}(t)\}$ $(k = 0, 1, \dots, n-1)$ converges uniformly on every segment contained in $(0, +\infty)$.

It is not hard to prove that $u(t) = \lim\limits_{i\to\infty} u_{m_i}(t)$ is a solution of equation (1), defined on the interval $(0, +\infty)$, and, on the other hand, it follows directly from (13) and (15) that $u(t)$ satisfies conditions (2).

Corollary 1. *Suppose that conditions (3) are satisfied, and that in the domain (4) inequality (5) holds, where $\omega(t, x) \ge 0$ for $t \in (0, a)$, $x \ge 0$, $\omega(t, x) \in K_t(a, a)$ for any $a \in (0, a)$, and there exists a positive number ρ_0 such that the upper solution $\rho(t)$ of the problem*

$$d\rho/dt = -\omega(t, \rho), \qquad \rho(a) = \rho_0$$

is defined on the interval $(0, a]$ and

$$(n-2)!\, r < \int_0^a t^{n-2} \rho(t)\, dt < +\infty.$$

Then problem (1)–(2) is solvable for any $u_0 \in [0, r]$.

The condition of Corollary 1 is satisfied, for example, by the function $\omega(t, x) = \psi(t)\omega(x)$, if $\psi(t) > 0$ for $t \in (0, a)$, $\omega(x)$ is positive and continuous on the interval $(0, +\infty)$, and either

$$\int_0^a \psi(t)\,dt < \int_0^{+\infty} \frac{dt}{\omega(t)} < +\infty, \qquad \int_0^a t^{n-2}\Omega^{-1}\left[\int_0^t \psi(\tau)\,d\tau\right] dt > (n-2)!\,r,$$

where $\Omega^{-1}(t)$ is the function inverse to

$$\Omega(t) = \int_t^{+\infty} \frac{d\tau}{\omega(\tau)},$$

or

$$\Omega_\delta(t) = \int_\delta^t \frac{d\tau}{\omega(\tau)} \to +\infty \quad \text{as } t \to +\infty, \qquad t^{n-2}\Omega_\delta^{-1}\left[\int_t^a \psi(\tau)\,d\tau\right] \in L(0,a),$$

where

$$\delta > (n-1)!\,a^{1-n}r,$$

and $\Omega_\delta^{-1}(t)$ is the function inverse to $\Omega_\delta(t)$.

From Corollary 1 one easily obtains the following simple

Corollary 2. *Suppose that conditions (2) are satisfied, $a > 0$, $r > 0$, and in the domain $t \in (0,a)$, $(-1)^{k-1}x_k \geq 0$ $(k = 1, 2, \dots, n)$, we have*

$$|f(t, x_1, \dots, x_n)| \leq At^{(n-1)\lambda-n}|\ln t|^{\lambda-1-\varepsilon}\sum_{k=2}^{n}(1+|x_n|)^{\mu_k}(1+|x_k|)^{\frac{n-1}{k-1}\nu_k},$$

where $A \geq 0$, $\nu_k \geq 0$, $\mu_k + \nu_k \leq \lambda$ $(k = 2, \dots, n)$, $\varepsilon > 0$ for $\lambda \leq 1$ and $\varepsilon = 0$ for $\lambda > 1$. Then problem (1)–(2) is solvable for any $u_0 \in [0, r]$.

It should be noted that the above-stated conditions of the form (5) are, in a certain sense, close to necessary conditions. As examples, let us consider the equations

$$u^{(n)} = (-1)^n t^{(n-1)\lambda-n}(1+|\ln t|)^{\nu}(1+|u^{(n-1)}|)^{\lambda}u; \tag{16}$$

$$u^{(n)} = (-1)^n t^{(n-1)\mu-n}(1+|\ln t|)^{\eta}(u+|u^{(n-1)}|)^{\mu}, \tag{17}$$

where $\lambda \leq 1$, $\mu > 1$. It follows directly from Corollary 2 of Theorem 1 that, if $\nu < \lambda - 1$ and $\eta \leq \mu - 1$, then, for any $u_0 \geq 0$, equations (16) and (17) have solutions satisfying conditions (2); on the other hand, one can show that if $\nu \geq \lambda - 1$, then problem (16)–(2) has no solution for any $u_0 > 0$, while if $\eta > \mu - 1$, then problem (17)–(2) has no solution for sufficiently large positive u_0.

Theorem 2. *If $f(t, x_1, \dots, x_n) \in K_t(0, a)$ for any $a \in (0, +\infty)$ and conditions (3) are satisfied, then there exists a positive number r such that problem (1)–(2) is solvable for any $u_0 \in [0, r]$.*

In conclusion we give two theorems on the behavior of solutions of problem (1)–(2) as $t \to +\infty$.

Theorem 3. Suppose that conditions (3) hold, $a > 0$, $r > 0$, $\varepsilon > 0$, and in the domain $t \in [a, +\infty)$, $0 \leq x_1 \leq r$, $0 \leq (-1)^{k-1} x_k \leq \varepsilon$ $(k = 2, \dots, n)$ we have

$$|f(t, x_1, \dots, x_n)| \geq \sigma_1(t, x_1) + \sigma_2(t, x_1)|x_k|,$$

where $\sigma_k(t, x_1)$ $(k = 1, 2)$ are functions nondecreasing in x_1, $\sigma_k(t, x_1) \in K_t(0, \alpha)$ for every $\alpha \in (a, +\infty)$ $(k = 1, 2)$, and for every $x_1 \in (0, r]$

$$\lim_{s \to +\infty} \int_a^s t^{n-2} \left\{ \int_t^s \sigma_1(\tau, x_1) \exp\left[\int_t^\tau \sigma_2(\xi, x_1)\, d\xi \right] d\tau \right\} dt = +\infty.$$

Then, whatever $u_0 \in [0, r]$, every solution $u(t)$ of problem (1)–(2) satisfies the conditions

$$\lim_{t \to +\infty} t^k u^{(k)}(t) = 0 \qquad (k = 0, 1, \dots, n-1).$$

Theorem 4. Suppose that conditions (3) hold, $r > 0$, and, for any $x > 0$, $a > 0$, the function $f(t, x, 0, \dots, 0)$ is different from zero on a set of positive measure of the interval $[a, +\infty)$, and for sufficiently large t, in the domain (4), we have

$$|f(t, x_1, \dots, x_n)| \geq \delta t^{(k-1)\lambda - n} |x_k|^\lambda,$$

where $1 \leq k \leq n$, $\delta > 0$, $0 < \lambda < 1$. Then, whatever $u_0 \in [0, r]$, for every solution $u(t)$ of problem (1)–(2) there exists a number t_0 such that

$$u(t) \equiv 0 \quad \text{for } t \geq t_0.$$

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Note: Figure translations are in progress. See original paper for figures.

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