

# MINIMAXITY OF HOTELLING' S $(T^2)$ -TEST

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## Abstract

## Full Text

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MATHEMATICS

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## MINIMAXITY OF HOTELLING' S $T^2$ -TEST

(Presented by Academician Yu. V. Linnik on 20 III 1968)

The problem considered here is as follows. Let  $X_1, \dots, X_N$  be a repeated sample from a  $p$ -dimensional normal population characterized by a mean vector  $\xi$  and covariance matrix  $\Sigma$ . The hypothesis  $H_0 : \xi = 0$  is tested against the alternative  $H_\delta : N\xi^T\Sigma^{-1}\xi = \delta$ , where  $\delta > 0$  is a given number. Among randomized tests  $\varphi$  of level  $\alpha \in (0, 1)$ , it is required to find a test  $\varphi_0$  that would satisfy the condition

$$\sup_{\varphi} \inf_{(\xi, \Sigma) \in H_\delta} E_{\xi, \Sigma} \varphi = \inf_{(\xi, \Sigma) \in H_\delta} E_{\xi, \Sigma} \varphi_0, \quad (1)$$

i.e., would maximize the minimum power on the alternative  $H_\delta$ .

The statistical meaning of condition (1) is clear.

Of course, the problem formulated is not the only one of its kind. There are other classical situations in multivariate analysis where the question is to find tests with guaranteed power. However, a common characteristic feature of this circle of problems is the considerable analytical difficulty and the absence of definitive results.

Let us return to our problem. Put

$$N\bar{X} = \sum_{i=1}^N X_i, \quad S = \sum_{i=1}^N (X_i - \bar{X})(X_i - \bar{X})^T.$$

To test  $H_0$  against  $H_\delta$ , Hotelling' s  $T^2$ -test is usually applied; it rejects  $H_0$  when

$$T^2 = N(N-1)\bar{X}^T S^{-1} \bar{X} > T_0^2,$$

where  $T_0^2$  is a constant determined by the level  $\alpha$ . The statistical properties of the  $T^2$ -test were studied in (1-4). The thought arose that this test might also be minimax. This thought was considerably strengthened after the report appearing in (5), which rejected it, turned out to be erroneous; soon in (6) the minimaxity of Hotelling' s  $T^2$ -test was proved for  $p = 2$ ,  $N = 3$ . It was shown that the minimax test of the hypothesis  $H_0$  against  $H_\delta$  may be sought in the class

of tests (depending on the sufficient statistics  $\bar{X}$  and  $S$ ) that are invariant with respect to the group of transformations formed by nonsingular lower triangular matrices (zeros above the main diagonal). Of course, the  $T^2$ -test is invariant with respect to this group. It was then observed that the  $T^2$ -test is minimax in the class of invariants of the indicated group if and only if it is Bayes. The attempt to express the Bayesness of the test analytically leads in a natural way to a Fredholm integral equation of the first kind. In the present case it has the form

$$\int_{\Gamma} \exp \sum_{i>j} \frac{t_j \beta_i}{2} \cdot \prod_{i=1}^p \Phi \left( \frac{N-i+1}{2}, \frac{1}{2}, \frac{t_i \beta_i}{2} \right) d\lambda^*(\beta_1, \dots, \beta_p) = \Phi \left( \frac{N}{2}, \frac{p}{2}, \frac{\gamma}{2} \right), \quad (2)$$

where  $\Gamma$  denotes the  $(p-1)$ -dimensional simplex

$$\left\{ (\beta_1, \dots, \beta_p) : \beta_i \geq 0, \sum_{i=1}^p \beta_i = 1 \right\}.$$

and  $\Phi$  is the degenerate hypergeometric function. Equation (2) must hold identically in  $t_1, \dots, t_p$  satisfying the condition

$$\sum_{i=1}^p t_i = \gamma, \quad \gamma > 0$$

where  $\gamma$  is any fixed number. The desired function  $\lambda^*$  must be a probability measure on  $\Gamma$ .

Following the result of N. Giri, J. Kiefer, and C. Stein <sup>(6)</sup>, on the basis of a new approach to the solution of the corresponding Fredholm equation of the first kind, in <sup>(7)</sup> the minimaxity of the  $T^2$ -test was proved for  $p = 2$ ,  $N = 4$ . In that work the specific nature of the equation was, in a certain sense, correctly used. Recognition of this allowed the authors to express confidence that consideration of the general case would require only a complication of the arguments carried out. At present this confidence can be assessed, although its realization required a rather long time.

At the same time, Yu. V. Linnik <sup>(8)</sup> introduced the notion of strong approximate minimaxity. The essence of the matter consisted in a certain weakening of condition (1) while preserving its basic content. The study of tests of multivariate analysis from the point of view of properties close to minimaxity was thereby facilitated. A number of positive results were obtained. But already the investigation <sup>(9)</sup> showed that this is connected with very complicated analytic constructions.

Let us note the work <sup>(10)</sup>, where the minimaxity problem is posed and, in the simplest case, solved for the complex analogue of the  $T^2$ -test (in the case of a sample of size  $N$  from a  $p$ -dimensional normal complex distribution). On the basis of <sup>(6)</sup> the problem is reduced to the following Fredholm integral equation of the first kind:

$$\int_{\Gamma} \exp \sum_{i>j} t_j \beta_i \cdot \prod_{i=1}^p \Phi(N-i+1, 1, t_i \beta_i) d\lambda^*(\beta_1, \dots, \beta_p) = \Phi(N, p, \gamma), \quad (3)$$

where the meanings of the symbols and the conditions on them are the same as for equation (2).

In the present article we present the answer to the question of the minimax nature of the  $T^2$ -test.

**Theorem 1.** *Hotelling's  $T^2$ -test of the hypothesis  $H_0$  against  $H_{\delta}$  (as well as its complex analogue) is minimax for any  $N > p \geq 2$  and any level  $\alpha \in (0, 1)$ .*

If instead of  $H_{\delta}$  we take the alternative  $H'_{\delta} : N\xi' \Sigma^{-1} \xi \geq \delta$ , then we arrive at a problem which should be regarded as more natural.

**Theorem 2.** *Hotelling's  $T^2$ -test of the hypothesis  $H_0$  against  $H'_{\delta}$  (as well as its complex analogue) is minimax for any  $N > p \geq 2$  and any level  $\alpha \in (0, 1)$ .*

Theorem 2 is closely connected with Theorem 1.

Below we shall explain, in the roughest outline, the essence of the proof of Theorem 1 for  $p = 2$ . Putting in this case  $\lambda(\beta_2) = d\lambda^* \Pi^{-1}(\beta_2) / d\beta_2$ , where  $\Pi$  is the projection mapping of the simplex  $\Gamma$  onto the  $\beta_2$ -axis,  $\beta_2 = x$ , we obtain from (2)

$$\int e^{(a-t)x} \Phi\left(\frac{N}{2}, \frac{1}{2}, (a-t)(1-x)\right) \Phi\left(\frac{N-1}{2}, \frac{1}{2}, tx\right) \lambda(x) dx = \Phi\left(\frac{N}{2}, 1, a\right) \quad (4)$$

where  $\gamma = 2a$ ,  $t_2 = 2t$ , and from (3) we obtain

$$\int_0^1 e^{(a-t)x} \Phi(N, 1, (a-t)(1-x)) \Phi(N-1, 1, tx) \lambda(x) dx = \Phi(N, 2, a), \quad (5)$$

where  $\gamma = a$ ,  $t_2 = t$ .

For the solution of the problem in the case  $p = 2$ , it is sufficient that there exist  $\lambda(x)$  which is a probability density on  $[0, 1]$ .

**Theorem 3.** Let  $\gamma > 0$ . The integral equation

$$\int_0^1 e^{(a-t)x} \Phi(\alpha + \gamma, \gamma, (a-t)(1-x)) \Phi(\alpha, \gamma, tx) \lambda(x) dx = \Phi(\alpha + \gamma, 2\gamma, a) \quad (6)$$

in the cases

a)  $\alpha = \gamma + n, n = 1, 2, \dots, \gamma > 0;$

b)  $\alpha = n, n = 1, 2, \dots, 0 < \gamma < 1$

has a solution in the form of a probability density on the interval  $[0, 1]$ .

Taking the first case of this theorem, we obtain equation (4) for an even sample size  $N$ , putting  $\gamma = \frac{1}{2}, N = 2(1+n)$ , and equation (5), putting  $\gamma = 1, N = 2+n$ . Taking the second case, we obtain equation (4) for an odd sample size  $N$ , putting  $\gamma = \frac{1}{2}, N = 1 + 2n$ .

Thus, Theorem 1 follows from Theorem 3.

Introduce the notation

$$A_\alpha = e^{(-t)x} \Phi(\alpha + \gamma, \gamma, (a-t)(1-x)) \Phi(\alpha, \gamma, tx),$$

$$L(f) = \frac{1}{\gamma} x(1-x) f'' + \left[ (1-2x) - \frac{1}{\gamma} ax(1-x) \right] f'.$$

The basis of the proof of Theorem 3 is the functional relation

$$L(A_\alpha) = -\frac{\alpha(\alpha + \gamma)}{\gamma} A_{\alpha+1} + \frac{\alpha(2\alpha + a)}{\gamma} A_\alpha - \frac{\alpha(\alpha - \gamma)}{\gamma} A_{\alpha-1},$$

which makes it possible in both cases a) and b) to represent the kernel of equation (6) in several different ways as a second-order linear differential operator applied to the corresponding function. “Transferring” this operator to  $\lambda_i(x)$  (the index  $i$  corresponds to the chosen method of representation), we obtain a differential equation for it, homogeneous in case a) and inhomogeneous in case b) (this is where the difference between the cases appears). The equation fits within the theory of linear differential equations with regular singular points. The behavior of the properly constructed solution  $\lambda_i(x)$  at the endpoints of the interval  $(0, 1)$  makes it possible to compute the nonintegral term arising in the aforementioned “transfer” of the operator. Forming a linear combination of the obtained functions  $\lambda_i(x)$  and substituting it into (6), one can arrange that the right-hand side be equal to a constant. This constant is found uniquely, and its value coincides with the right-hand side of (6). The investigation of the sign of the linear combination of the functions  $\lambda_i(x)$  is connected with the study of second-order linear equations, which determines its success.

We note that, in reality, the proof program for Theorem 3 described above is implemented rather laboriously. The proof of Theorem 1 for  $p > 2$  is accompanied by even greater bulkiness and complexity.

The conceptual side of the solution of the problem considered was heuristically formulated by the author in February 1967 in Berlin in a report at a meeting of the Probability Theory and Mathematical Statistics section of the Fourth Congress of the Mathematical Society of the GDR.

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*Note: Figure translations are in progress. See original paper for figures.*

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